MATHEMATICS

PART - 2

Standard

IX

Government of Kerala
Department of General Education

Prepared by
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2024

THE NATIONAL ANTHEM

Jana-gana-mana adhinayaka, jaya he
Bharatha-bhagya-vidhata
Punjab-Sindh-Gujarat-Maratha
Dravida-Utkala-Banga
Vindhya-Himachala-Yamuna-Ganga
Uchchala-Jaladhi-taranga
Tava subha name jage,
Tava subha asisa mage,
Gahe tava jaya gatha
Jana-gana-mangala-dayaka jaya he
Bharatha-bhagya-vidhata
Jaya he, jaya he, jaya he,
Jaya jaya jaya, jaya he.

PLEDGE

India is my country. All Indians are my brothers and sisters.

I love my country, and I am proud of its rich and varied heritage. I shall always strive to be worthy of it.

I shall give my parents, teachers and all elders, respect and treat everyone with courtesy.

To my country and my people, I pledge my devotion. In their well-being and prosperity alone, lies my happiness.

MATHEMATICS

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Dear children,

To understand the world and to recognise its peculiarities, all kinds of measurements and the relations between them were essential for humans.

We've seen how natural numbers and fractions, and illustrations with them evolved like this, in earlier classes. Here we will see certain measures which cannot be expressed using numbers learnt so far.

We also continue our study of geometry. This book discusses the new idea of similarity and how it ties up with the notion of parallel lines studied earlier.

Our new vision of learning mathematics emphasizes computational thinking, which gives greater emphasis to analysis of mathematical processes. What this means is that, instead of learning different mathematical techniques by rote and using them mechanically in specified contexts, it is more important to understand how and why they work and to use different methods to make them more efficient.

We also discuss the possibilities of the free software GeoGebra to make geometry more dynamic and thereby enhance comprehension.

With love and regards,

Dr. Jayaprakash R.K.
Director
SCERT Kerala

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Certain icons are used in this textbook for convenience


Let's do problems


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THE CONSTITUTION OF INDIA

PREAMBLE

WE, THE PEOPLE OF INDIA, having solemnly resolved to constitute India into a [1]**[SOVEREIGN SOCIALIST SECULAR DEMOCRATIC REPUBLIC]** and to secure to all its citizens :

JUSTICE, social, economic and political;

LIBERTY of thought, expression, belief, faith and worship;

EQUALITY of status and of opportunity; and to promote among them all

FRATERNITY assuring the dignity of the individual and the [2][unity and integrity of the Nation];

IN OUR CONSTITUENT ASSEMBLY this twenty-sixth day of November, 1949 do **HEREBY ADOPT, ENACT AND GIVE TO OURSELVES THIS CONSTITUTION.**

---

1. Subs. by the Constitution (Forty-second Amendment) Act, 1976, Sec.2, for "Sovereign Democratic Republic" (w.e.f. 3.1.1977)
2. Subs. by the Constitution (Forty-second Amendment) Act, 1976, Sec.2, for "Unity of the Nation" (w.e.f. 3.1.1977)

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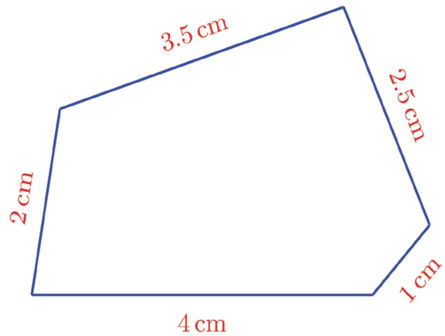
CIRCLE MEASURES

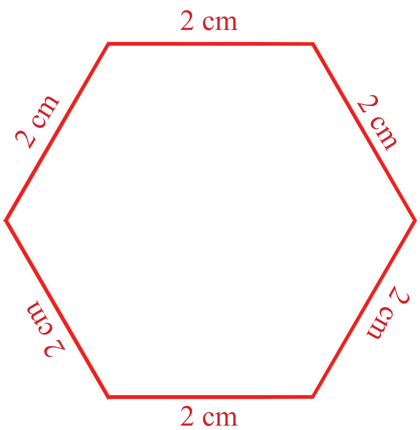
Diameter and perimeter

It's easy to calculate the perimeter of a polygon; just add the lengths of sides:



Perimeter 4 + 1 + 2.5 + 3.5 + 2 = 13 centimetres

All the more easy if the polygon is regular:



Perimeter 6 × 2 = 12 centimetres

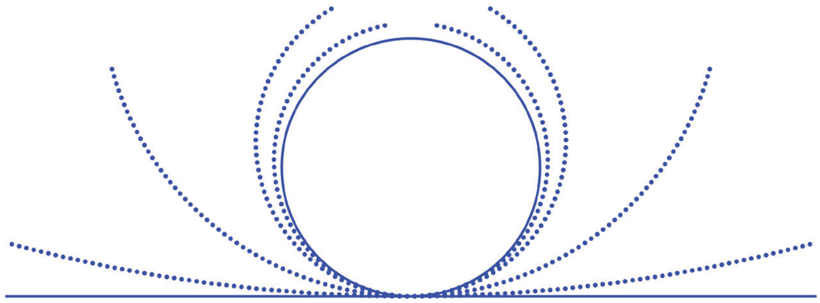
What about a circle?

We can use a piece of string or thread to measure the perimeter. But there are instances where this is not possible.

For example, see this problem:

A piece of wire is to be bent into a circle of diameter 5 centimetres. What should be the length of the wire?

When the wire is bent into a circle, the length of the wire becomes the perimeter of the circle:



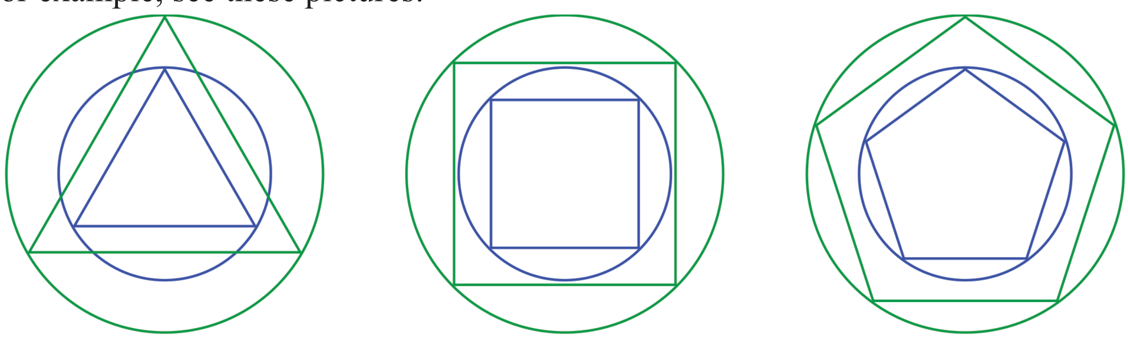
So to solve the problem above, we must know the relation between the diameter and perimeter of the circle.

The perimeter of a circle is often called its *circumference*. Evidently as the diameter is made larger, the circumference also becomes larger. The question we ask next is whether these changes are in the same scale; that is, if the diameter is changed to a fraction or multiple of the original, would the circumference also change to the same fraction or multiple of the original?

It is not difficult to see that the perimeters of regular polygons drawn within circles of different diameters are scaled by the same factor as the diameters of the circles.

We have seen in the lesson, **Similar Triangles** that the sides of such regular polygons are scaled by the same factor as the radii.

For example, see these pictures:



Regular polygons are drawn within two circles, the larger circle having a radius one and a half times that of the smaller. In each picture, the sides of the larger regular polygon is also one and a half times the sides of the smaller polygon.

Since diameter is double the radius, the diameter of the larger circle also is one and a half times the diameter of the smaller one.

Again, the perimeter of each polygon is a fixed multiple of the side (three, four and five for the different polygons in the pictures). So, perimeters of the larger polygons are also one and a half times those of the smaller ones.

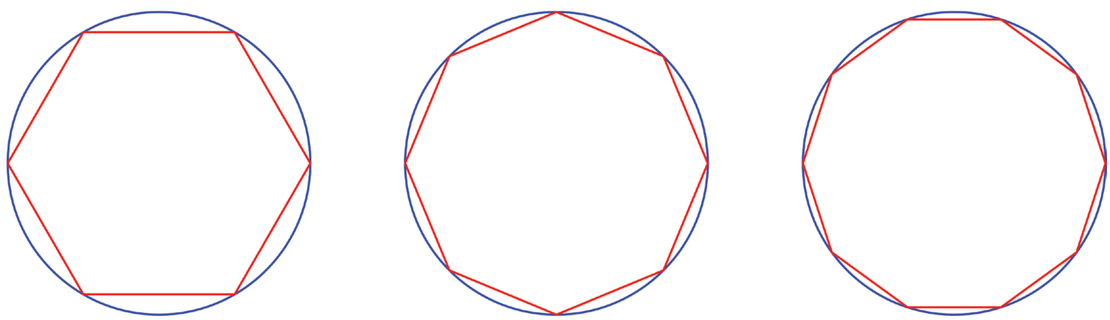
Thus, the perimeters of the polygons are scaled by the same factor as the diameters of the circles.

From this, can we say that the circumferences (perimeters) of the circles also are scaled by the same factor as the diameters?

For example, can we say that if the diameter of a circle is doubled, then the circumference also is doubled?

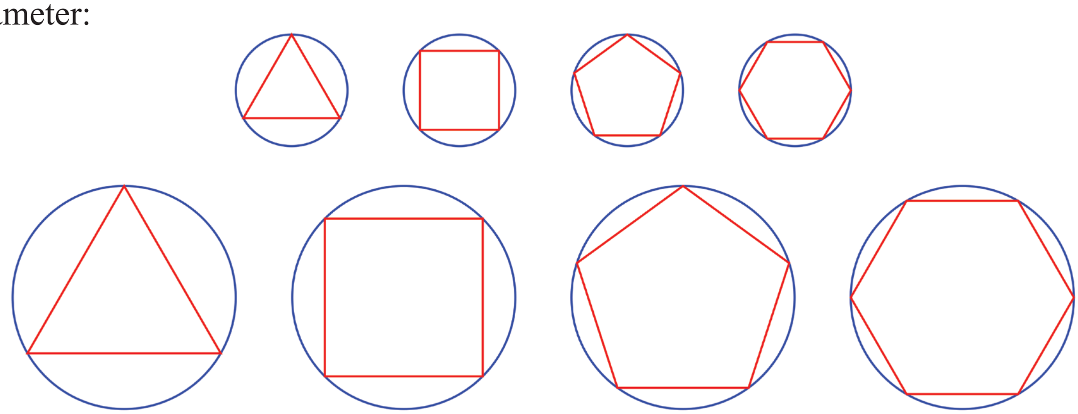
> Draw a circle in GeoGebra, centered at a point A and mark a point B on it. Create a slider n, selecting the option Integer, with Min : 3 and Max : 300. Select the Angle with Given Size tool and click on B and then on A. In the dialog window opening up, give the Angle as (360/n)°. Select the Regular Polygon tool and click on B and B′. In the dialog window, give n as the number of vertices. This will give a regular *n*-gon within the circle. Increase the number of sides using the slider. What happens to the polygons?

Not yet. We have to note another thing before we can claim this. As we keep on drawing regular polygons of more and more sides within a circle, they get closer and closer to the circle. See these pictures:



6 sides 8 sides 10 sides

Now suppose such polygons are drawn within a circle and another circle of double the diameter:



Since the diameter of the larger circle is twice the diameter of the smaller one, the perimeter of each large polygon is double the perimeter of the smaller one.

The perimeter of polygons in both circles gets closer and closer to the circumference of the circles.

We use a bit of algebra to make the relation between the measures clear:

- The perimeters of the polygons within the larger circle are twice the perimeters of the polygons within the smaller circle
  - ♦ Let's denote the perimeter of the smaller equilateral triangle by p_1, that of the smaller square by p_2, that of the smaller pentagon as p_3 and so on.
  - ♦ The perimeters of the larger polygons are $2p_1, 2p_2, 2p_3, \dots$
- The perimeters of the small polygons get closer and closer to the circumference of the small circle
  - ♦ Let's denote the circumference of the small circle by c
  - ♦ The numbers $p_1, p_2, p_3, \dots$ get closer and closer to c
- The perimeters of the large polygons get closer and closer to the circumference of the large circle
  - ♦ Let's denote the circumference of the large circle by d
  - ♦ The numbers $2p_1, 2p_2, 2p_3, \dots$get closer and closer to d

Note the two statements in the red boxes above. The first of these say that the numbers p_1, p_2, p_3, ... get closer and closer to c

So, the numbers $2p_1$, $2p_2$, $2p_3$, ... must get closer and closer to $2c$

But the second statement says that the numbers $2p_1, 2p_2, 2p_3$, ... must get closer and closer to d

This means $d = 2c$, doesn't it?

Thus the circumference of the larger circle is twice the circumference of the smaller circle. In the same way, we can see that if the diameter of a circle is scaled by some other factor, then the circumference also is scaled by the same factor

Create a slider d in GeoGebra with Min:0 and Max:10. Next, draw a circle of diameter d using the Circle:Center & Radius tool, give the radius as $\frac{d}{2}$. Selecting the Distance or Length and clicking on the circle gives its perimeter.It can be seen in the Algebra window also. Right click on this and select the Rename option to change its name to c. Change the diameter of the circle using the slider and see whether the diameter and circumference are scaled by the same factor. Type $\frac{c}{d}$ in the Input Bar to get the quotient of the circumference by diameter in the Algebra window. Does this change when the diameter is changed?

In general, we have the following:

The circumference of circles are scaled by the same factor as their diameters

So, if we determine the circumference of a circle of diameter 1, then the circumference of any circle can be calculated by multiplying the diameter by this number.

Before doing this, try these problems, based on the facts seen so far:

(1) The circumference of a circle of diameter 2metres is measured and found to be 6.28 metres.

(i) How do we compute the circumference of a circle of diameter 4 metres, without actually measuring it?

(ii) What about the circumference of a circle of diameter 1metre?

(iii) And the circumference of a circle of diameter 3metre?

(2) A piece of wire is bent into a circle of diameter 4 centimetres. If a wire of half the length is bent into a circle, what would be its diameter ?

A new number

The circumference of a circle of diameter 1 cannot be written as a fraction, like the diagonal of a square of side 1 or the height of an equilateral triangle of side 2.

It is not easy to prove this, as in the case of other lengths. It was proved only in the eighteenth century.

There is a basic difference between this number and numbers like $\sqrt{2}, \sqrt{3}, \sqrt{2}+\sqrt{3}$ and so on: it cannot be written in terms of roots of natural numbers or fractions, and their basic operations.

A special symbol is used in mathematics to denote it: π

It is the letter called *pi* in the Greek alphabet.

Since π is not a fraction, we can only calculate fractions approximately equal to it. Up to four decimal places, it is given by

$$\pi \approx 3.1416$$

Thus the circumference of a circle of diameter 1 centimetre is π centimetres, the circumference of a circle of diameter 2 centimetres is 2π centimetres, the circumference of a circle of diameter $1\frac{1}{2}$ centimetres is $\frac{3}{2}\pi$ centimetres and so on

This is the general result:

The circumference of a circle is π times its diameter.

Since circles are mostly drawn with a specified radius, this is often stated in terms of the radius:

The circumference of a circle is 2π times its radius.

In the third century BCE, the Greek mathematician Archimedes computed that the circumference of a circle is more than $3\frac{10}{71}$ of the diameter and less than $3\frac{1}{7}$ of the diameter, using a regular polygon of 96 sides. In modern terminology, this translates to

$$3.1408 < \pi < 3.1428$$

up to four decimal places (Archimedes' estimate of $\frac{22}{7}$ was used to compute the circumference of a circle for a long time).

In the fourteenth century CE, the Kerala mathematician Madhavan developed a method to compute π to any desired degree of accuracy by purely numerical methods, without using geometry; for example, we can compute

$$\pi = 3.1415926535...$$

Now using computers, we can compute π up to billions of decimal places.

In practical problems, we usually require π only up to four decimal places. For example, the circumference of a circle of radius 5 metres correct up to millimetres is

$$\pi \times 2 \times 5 = 10\pi = 10 \times 3.1416$$
$$= 31.416 \text{ metres}$$

Draw a circle of diameter 1 in GeoGebra. What is its circumference? Selecting Options → Rounding → 15 Decimal Places, we can get π correct up to 15 decimal places

π in history

The efforts to compute the circumference and the area of circles started at least four thousand years ago. In current terminology, these can be interpreted as efforts to compute fractions approximating π

The solution of a problem done in the Ahmes Papyrus from ancient Egypt, interpreted this way, gives

$$\pi \approx \frac{256}{81} \approx 3.16$$

A Babylonian clay tablet from around the same period (BCE 1500) gives

$$\pi \approx \frac{25}{8} = 3.125$$

The *shatapathabrahmana* from India, estimated to be belonging to the seventh century BCE, this is given as $\frac{339}{108} \approx 3.139$

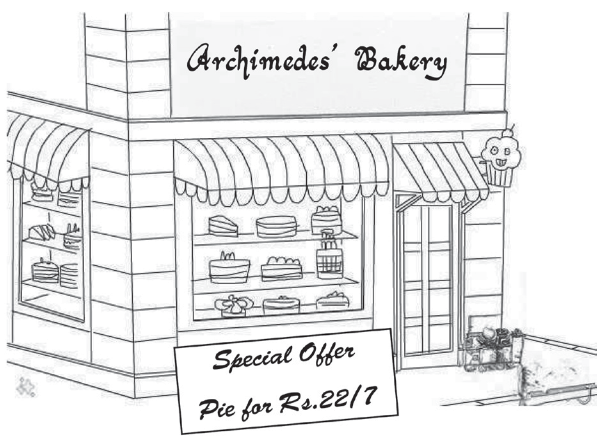
We have noted that Archimedes used 96 sided polygon to find $3\frac{10}{71} < \pi < 3\frac{1}{7}$ that is

$$3.1408 < \pi < 3.1428$$

In 180CE, Zu Chongzhi of China used a regular polygon of 12288 sides to find

$$3.1415926 < \pi < 3.1415927$$

and the smaller number is correct up to seven decimal places. Better approximations were found only after a thousand years



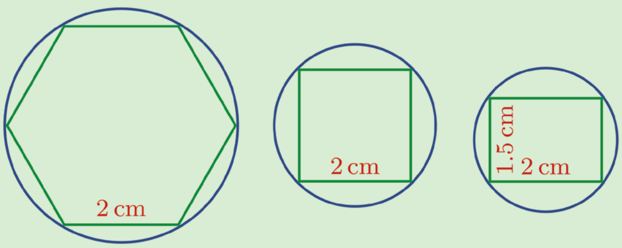
To make a circle of diameter 5 centimetres, we need a wire of length 5π centimetres; that is, approximately $5 \times 3.14 = 15.7$ centimetres.

Thus we have the answer to the problem posed at the beginning of the lesson.

Now try the problems below. In all these, you can write the measures using π itself; no need to use decimal approximations.

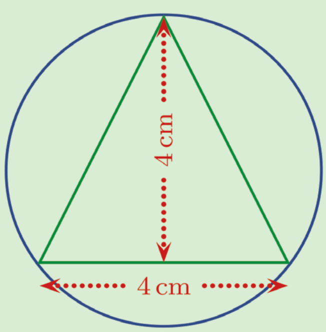
What is the circumference of a circle of radius $\frac{1}{2\pi}$ centimetres? To draw such a circle in GeoGebra, select the Circle Center & Radius tool and in the window opening on clicking at a point, give the radius as 1/(2pi). Now can you draw a circle of circumference 25 centimetres?

(1) Calculate the circumferences of the circles shown below:



(2) In a circle, a chord 4 centimetres away from the centre is 6 centimetres long. What is the circumference of the circle?

(3) The figure below shows an isosceles triangle of base and height 4 centimetres drawn with vertices on a circle:



Calculate the circumference of the circle.

π in Keralam

The method developed by the Kerala mathematician Madhavan (sometimes known as Samgamagrama Madhavan) to calculate fractions which approximate π is a turning point in the history of mathematics. His method is numerical, very much different from the geometrical methods used till then.

What he found was this: starting from 1, alternatively subtract and add reciprocals of the odd numbers, which gives the numbers

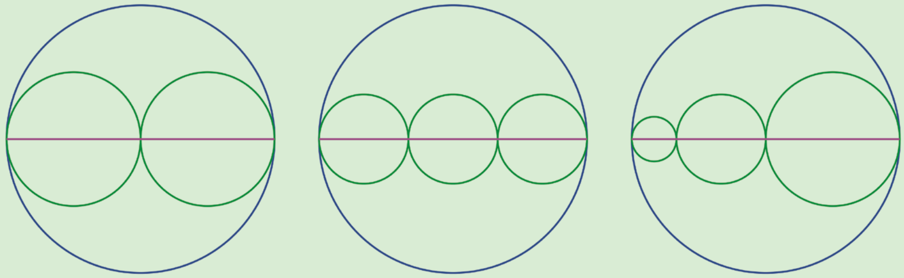
$$1, \quad 1-\frac{1}{3}, \quad 1-\frac{1}{3}+\frac{1}{5}, \quad 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}$$

and so on. These numbers get closer and closer to $\frac{1}{4}\pi$. This was rediscovered in the seventeenth century by Gregory of Scotland and Leibniz of Germany.

One defect of Madhavan's method is that these numbers approach $\frac{1}{4}\pi$ quite slowly. To get Archimedes' approximation, we will have to take up to 4000 of these numbers. But Madhavan himself discovered another method using which he could compute

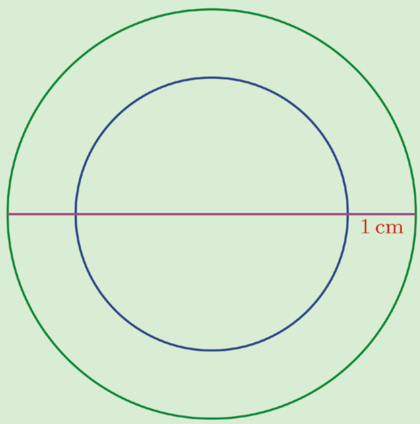
$$\pi \approx 3.14159265359$$

(4) In each of the pictures below, the centres of the large and small circles are on the same line. In the first and the second pictures, all the small circles have the same diameter:



In each of these figures, show that the circumference of the large circle is sum of the circumferences of the small circles.

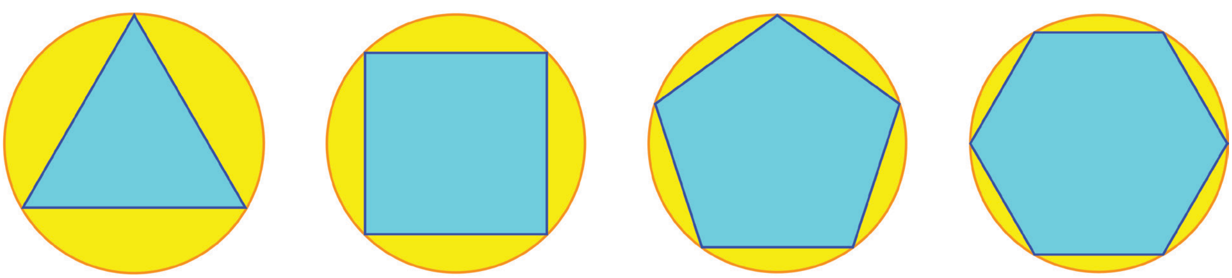
(5) In the picture below, the two circles have the same centre.



How much more is the circumference of the larger circle than that of the smaller circle?

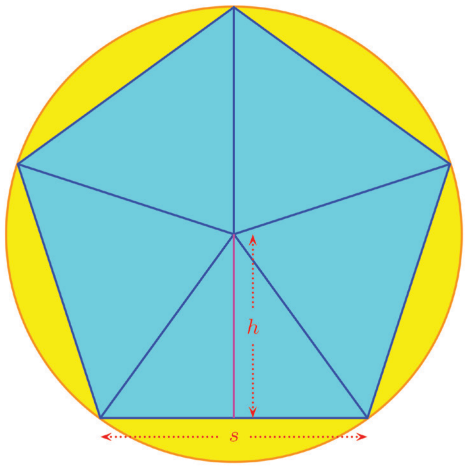
Area

Just as the perimeters of regular polygons within a circle approach the circumference of the circle, the areas of these polygons approach the area of the circle:



To calculate the area of the circle, we must see how the areas of these polygons increase. If we join the centre of the circle to the vertices of the polygon, we can split the polygon into equal triangles; and the area of the polygon is the sum of the areas of these triangles

For example, consider the regular pentagon inside the circle. If the length of its sides is denoted by s and the length of the perpendicular from the centre to one side is denoted by h, then the area of each of the five triangles is $\frac{1}{2}sh$:

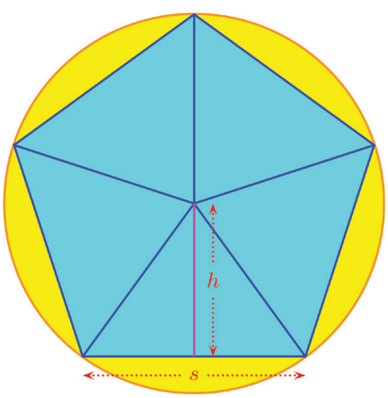


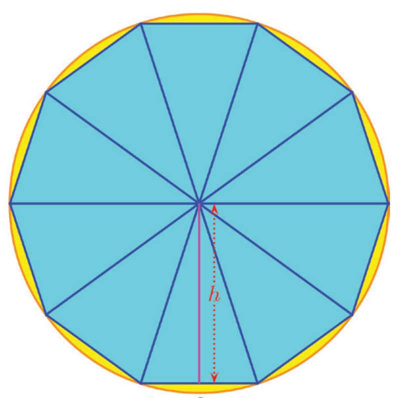
The pentagon is made up of five such triangles, so that its area is

$$5 \times \frac{1}{2}sh = \frac{1}{2} \times 5s \times h$$

In this, $5s$ is the perimeter of the pentagon, since s is the length of its sides. Denoting it by p, the area of the pentagon is $\frac{1}{2}ph$.

Whatever regular polygon we take inside the circle, its area can be found to be half the product of its perimeter and the length of the perpendicular from the center to a side. For different polygons, the perimeter and the length of the perpendicular are different:





As in the computation of the circumference, let's write these using algebra:

- We denote the perimeters of the equilateral triangle, square, regular pentagon and so on within the circle by $p_1, p_2, p_3, \ldots$ and the lengths of the perpendiculars from the centre to a side by $h_1, h_2, h_3, \ldots$

- Let's denote the radius of the circle by r and its circumference by c

  - The numbers $h_1, h_2, h_3, \ldots$ get closer and closer to r

  - The numbers $p_1, p_2, p_3, \ldots$ get closer and closer to c

- The numbers $\frac{1}{2}p_1h_1, \frac{1}{2}p_2h_2, \frac{1}{2}p_3h_3, \ldots$ get closer and closer to $\frac{1}{2}cr$

  - The numbers $\frac{1}{2}p_1h_1, \frac{1}{2}p_2h_2, \frac{1}{2}p_3h_3, \ldots$ are the areas of the polygons.

  - The areas of the polygons get closer and closer to the area of the circle

  - Let's denote the area of the circle by a.

- The numbers $\frac{1}{2}p_1h_1, \frac{1}{2}p_2h_2, \frac{1}{2}p_3h_3, \ldots$ get closer and closer to a

From the two statements above within red rectangles. we see that

$$a = \frac{1}{2}cr$$

What does this mean?

The area of a circle is half the product of its circumference and radius

We have seen that the circumference of a circle of radius r is $2\pi r$. So the area of this circle is

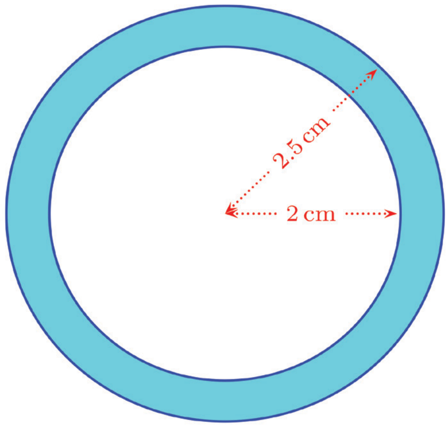
$$\frac{1}{2} \times 2\pi r \times r = \pi r^2$$

The area of a circle is π times the square of the radius

For example, the area of a circle of radius 5 centimetres is 25π square centimetres.

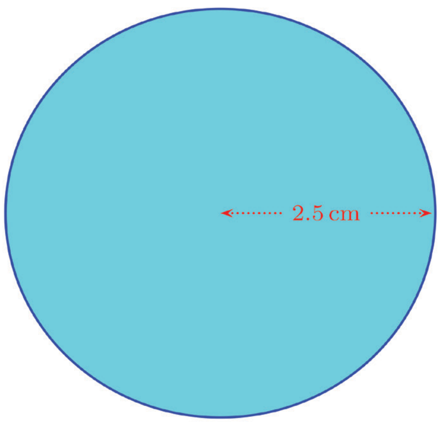
Now let's do some problems using this.

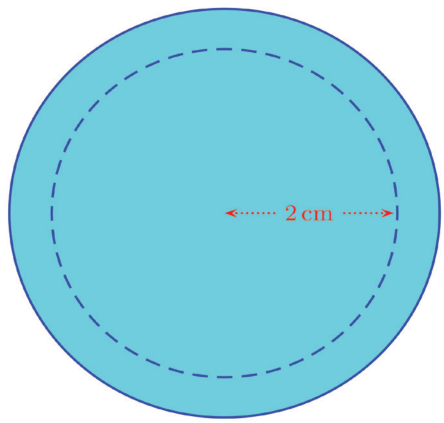
What is the area of the circular ring in this picture?



What is the area of a circle of radius $\sqrt{\frac{10}{\pi}}$? To draw such a circle in GeoGebra, just give the radius as sqrt (10/pi) in the dialoge window got from the Circle:Centre & Radius tool. On selecting the Area tool and clicking inside the circle, we get its area. Now can you draw a circle of area 25?

We can think of this as a circle cut off from a larger circle:

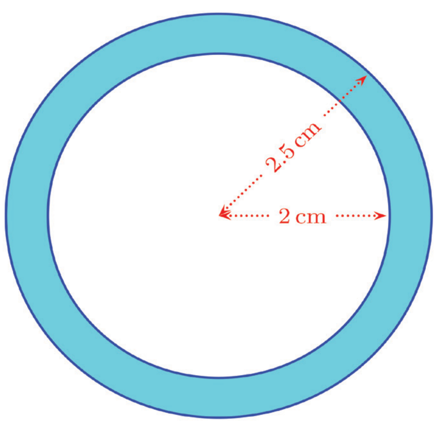


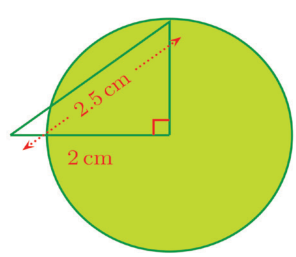


So, the area of the ring is

$$6.25\pi - 4\pi = 2.25\pi \text{ sq.cm}$$

Now let's draw a right triangle and a circle as below:

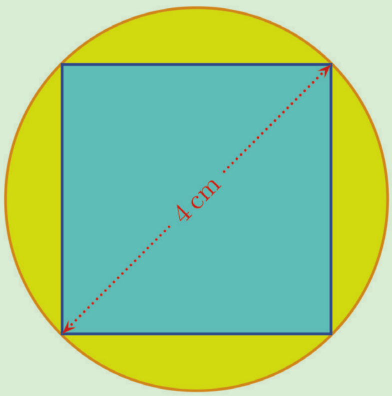


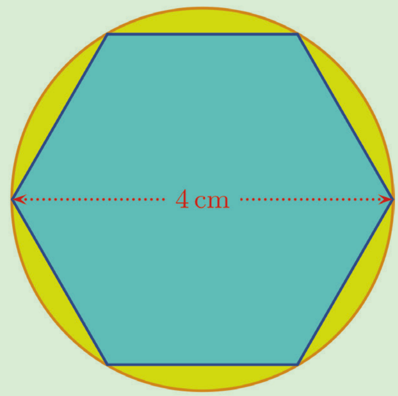


What is the relation between the areas of the blue ring and the green circle?

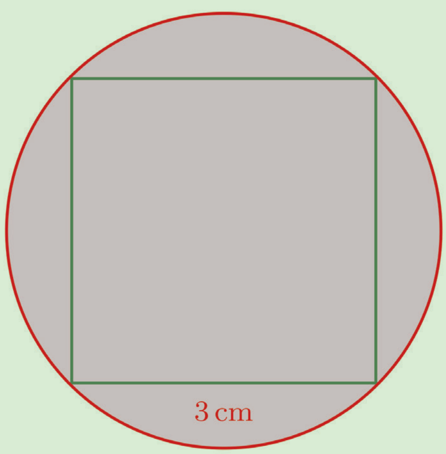
(1) The length of a chord of a circle, 3 centimetres from the centre, is 4 centimetres. What is the area of the circle?

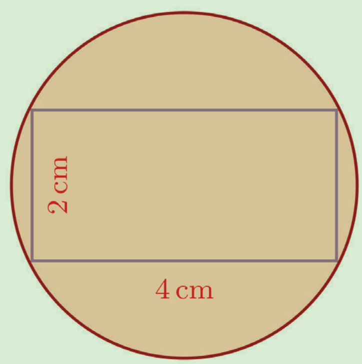
(2) In each of the pictures below, compute the difference between the area of the circle and the area of the regular polygon, correct up to two decimal places:





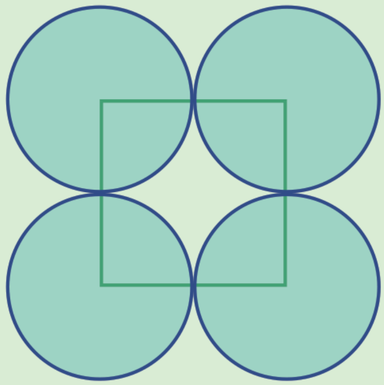
(3) In the pictures below, circles are drawn through the vertices of a square and a rectangle:

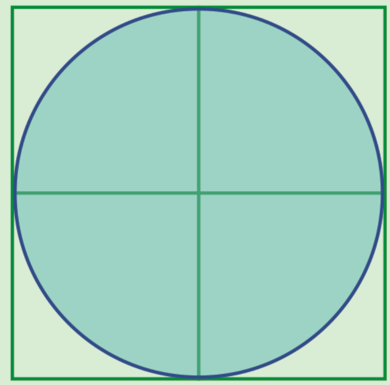




Calculate the areas of the circles

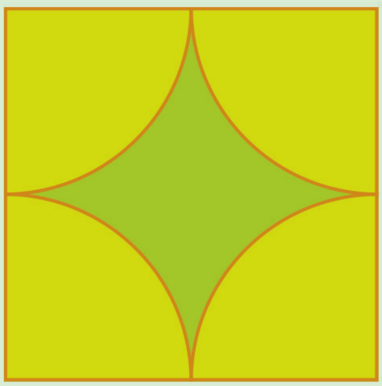
(4) Draw a square and draw circles with its vertices as centres and radius as half the side. Draw another square composed of four smaller squares of the same size as the first square, and draw a circle that just fits inside it.

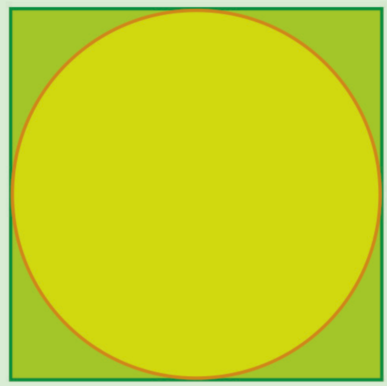




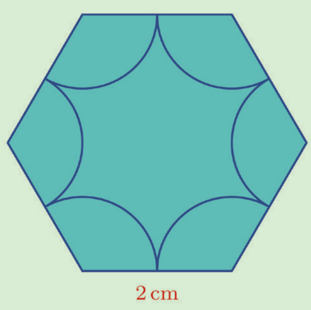
Prove that the area of the large circle is the sum of the areas of the four small circles.

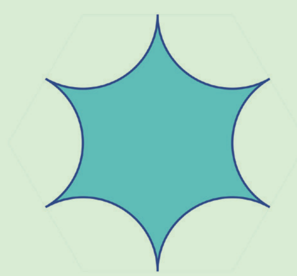
(5) In the pictures below, the squares are of the same size. Prove that the areas of the green regions in the pictures are equal:





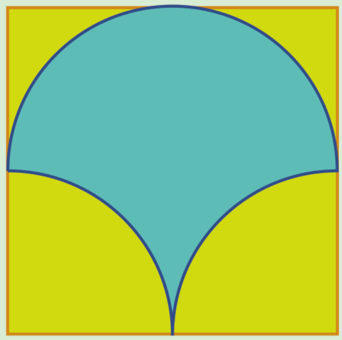
(6) Parts of circles are drawn with the vertices of a regular hexagon as centres and the figure below is cut out:





Calculate the area of the figure cut out.

(7) Parts of a circle are drawn within a square like this:



Prove that the area of blue region is half the area of the square.

9

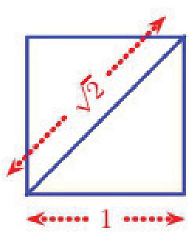
Real Numbers

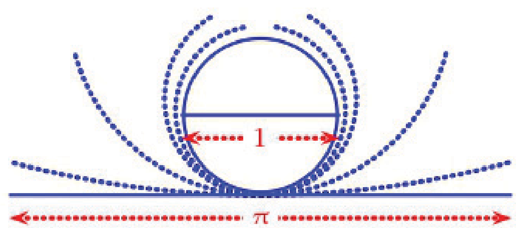
Measures and numbers

How do we express lengths of lines as numbers? If we take a definite length as 1, then twice this length can be termed 2, half this length $\frac{1}{2}$, one and a half times this length $1\frac{1}{2}$ and so on:

This length taken as 1 is called a *unit* of length. Once we specify a unit like this, many other lengths can be expressed as natural numbers or fractions, as seen above.

But there are some lengths which cannot be expressed as natural numbers or fractions of the chosen unit. For example, the diagonal of the square with length of sides this unit, or the circumference of the circle with diameter as this unit:





When relations between measures, and between operations on numbers, are expressed as algebraic equations, it is convenient to use negative numbers, as seen in the lesson, **Negative Numbers**. So, we also need the negatives $-\sqrt{2}$ and $-\pi$ of numbers like $\sqrt{2}$ and π.

Natural numbers, fractions, their negatives and zero are all collectively named *rational numbers*. All numbers which cannot be expressed as fractions are called *irrational numbers*. All numbers, rational and irrational numbers together are called *real numbers*:

We can write natural numbers also in fractional form: for example, 5 as $\frac{5}{1}$, $\frac{10}{2}$ or in so many other forms. Again, we can write negative natural number also in fractional form like this, with numerator or denominator negative.

Zero can be written as $\frac{0}{1}, \frac{0}{2}$ and so on.

Thus all rational numbers have a common form $\frac{x}{y}$; where x and y are natural numbers or their negatives; and x can be 0 also.

But among irrational numbers, there are roots like $\sqrt{2}$ or $\sqrt{3}$ and also numbers like π which cannot be expressed in terms of the usual operations on rational numbers. So, they cannot be confined to any common form.

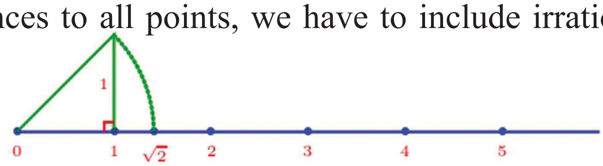
Numbers and geometry

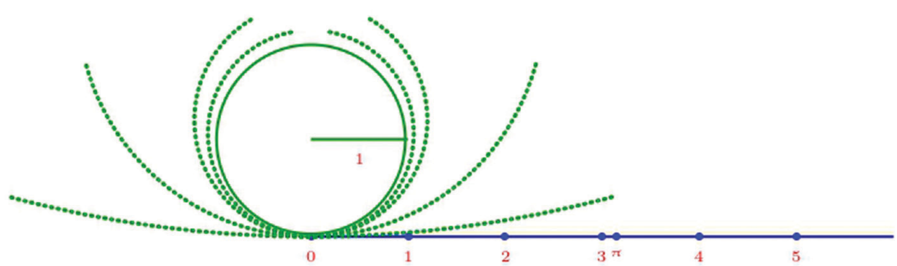
Every positive real number can be seen as a length. What if we consider all these lengths as starting from the same point?

To make this clear, think of a line, its left end point and another point to the right of this point:

Taking the distance between these points as the unit of length, we can write the distances to various points on the right as numbers:

If we want to mark the distances to all points, we have to include irrational numbers also:





We can extend this line to the left of the point marked 0; how do we mark points on this side with numbers?

For that, we can use the negatives of the numbers on the right. Recall how we did this in the section, **Position and number** of the lesson, **Negative Numbers**:

We can imagine all real numbers as points on this line; on the other hand, we can imagine all points on this line marked with real numbers

Such a line, with all points labelled using real numbers, is called a *number line* or *real line.*

On a number line, as we move to the right from 0, the numbers become larger. What if we move to the left?

Which is larger, −1 or −2?

−1 means 1 less than 0. What about −2? It is 2 less than 0, right? That is 1 less than −1. So −2 is smaller than −1 and we write

$$-2 < -1$$

Thus in a number line, we get larger numbers as we move right from 0, and smaller numbers as we move left from 0.

This is what happens if we move right or left from any number instead of zero, isn't it?

So, if we take any two real numbers, the position of the larger number on the number line would be on the right of the smaller number.

All kinds of numbers

Mathematics starts with the natural numbers 1, 2, 3, ..., Afterwards, for convenience in writing numbers (among other things), the number 0 began to be used. Fractions became necessary in the measurements of lengths, weights and time. Irrational numbers were needed since all lengths cannot be expressed as multiples or fractions of a single unit.

For convenience in using algebra, negatives of all these numbers began to be used. And all such numbers were termed real numbers.

Among real numbers, those which could be expressed as fractions were named rational numbers and the others, irrational numbers.

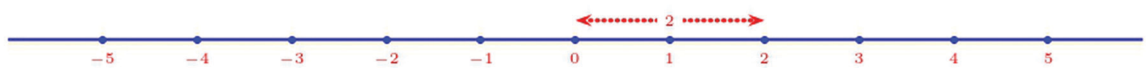
Natural numbers and their negatives, together with zero are called *integers.*

Thus the arithmetical relation of being smaller or larger translates to the geometrical relation of being on the left or right on the number line

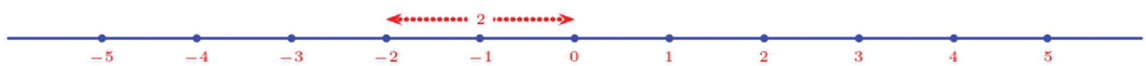
Now we see how the geometric representation of real number as points on a line leads to a new operation on numbers

Absolute value

We labelled points on a line using numbers, based on their distances from the point marked 0. For example, the distance between the points marked 0 and 2 is 2:



We marked as –2, the point at the same distance from 0, but to the left:



So, the distance from 0 to –2 is also 2

What can we say in general?

- The distance between zero and a positive number on the number line is that number itself
- The distance between zero and a negative number is the positive number got by removing the negative sign in that number

For example,

$$\text{The distance between 0 and } 5\tfrac{1}{2} \text{ is } 5\tfrac{1}{2}$$

$$\text{The distance between 0 and } -5\tfrac{1}{2} \text{ is } 5\tfrac{1}{2}$$

Now how do we use algebra to write the distance between the numbers 0 and x? (Note that in writing numbers as letters in algebra, we often write both positive and negative numbers as just x, y and so on, without attaching a sign).

So we have to describe the operation of removing the sign of a negative number in another way. Recall seeing in the lesson, **Negative Numbers** that the negative of the negative of a positive number is the number itself.

For example

$$-(-2) = 2$$

So instead of saying, "remove the sign of a negative number", we can say "take the negative of that number". Thus if x is a negative number, the positive number got by removing its negative sign is $-x$

For example if $x = -3$ then

$$-x = -(-3) = 3$$

Now we can say that the distance between 0 and a negative number x is $-x$

This operation of taking x itself if $x > 0$ (that is if x is a positive number) and $-x$ if $x < 0$ (that is, if x is a negative number) is written as $|x|$, and is called the *absolute value* of x.

For example,

$$|5| = 5 \qquad |-5| = 5$$

$$\left|\frac{2}{3}\right| = \frac{2}{3} \qquad \left|-\frac{2}{3}\right| = \frac{2}{3}$$

$$|\pi| = \pi \qquad |-\pi| = \pi$$

We take the absolute value of 0 as 0 itself.

The general description of absolute value can be stated like this:

If $x > 0$, then $|x| = x$

If $x < 0$, then $|x| = -x$

If $x = 0$, then $|x| = 0$

We can combine these into a single equation like this:

$$|x| = \begin{cases} x, & \text{if } x > 0 \\ -x, & \text{if } x < 0 \\ 0, & \text{if } x = 0 \end{cases}$$

We can summarize our discussion like this:

On the number line, the distance between the point labelled zero and the point labelled by another number is the absolute value of that number.

Using algebra, we can state it like this

On the number line, the distance between the point labelled 0 and the point labelled x is $|x|$.

Next we look at some properties of absolute values.

For example, what is the relation between the absolute values of a number and its negative?

For example, we have seen that

$$|3| = 3 = |-3|$$

What if we start with −3 instead? We have seen that $-(-3) = 3$. So, the absolute values of −3 and its negative are both equal to 3:

$$|-3| = |3| = |-(-3)|$$

This is true for all numbers:

Any number and its negative have the same absolute value

Using algebra, this can be stated like this:

$|-x| = |x|$ for any number x

Let's look at another fact.

The square of any number, positive or negative, is positive. For example,

$$5^2 = 5 \times 5 = 25$$
$$(-5)^2 = (-5) \times (-5) = 25$$

Now since

$$|5| = |-5| = 5$$

we have

$$|5|^2 = 5^2$$
$$|-5|^2 = 5^2 = (-5)^2$$

This is true whatever numbers we take. Thus, we have this general result

$|x|^2 = x^2$ for any number x.

(1) Complete the table below:

x	y	xy	$\lvert x\rvert$	$\lvert y\rvert$	$\lvert xy\rvert$	$\lvert x\rvert\,\lvert y\rvert$
4	3	12	4	3	12	12
−4	3					
4	−3					
−4	−3					

(i) Expand the table by taking some more pairs x, y of numbers. Do you see any relation between $|xy|$ and $|x|\,|y|$?

(ii) Prove that $|xy| = |x||y|$ for any two numbers x and y

(2) Complete the table below:

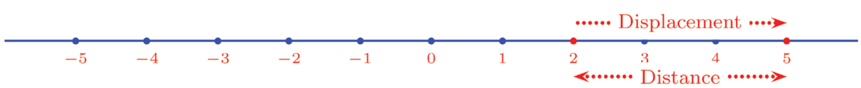
x	y	$x+y$	$\lvert x\rvert$	$\lvert y\rvert$	$\lvert x\rvert + \lvert y\rvert$	$\lvert x+y\rvert$
3	5	8	3	5	8	8
–3	5	2	3	5	8	2
3	–5					
–3	–5					

Expand the table by taking some more pairs x, y of numbers. Do you see any relation between $\lvert x\rvert + \lvert y\rvert$ and $\lvert x + y\rvert$?

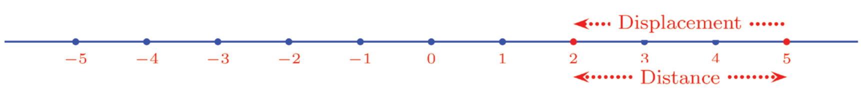
Distances

We have seen that the distance between the number 0 and the number x is $\lvert x\rvert$. Now let's see how we can write in algebra, the distance between two numbers x and y.

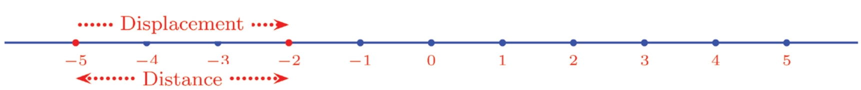
Recall how we calculated the displacement of a point moving along a line in the lesson, **Negative Numbers**: displacements to the right as just the distance, and displacements to the left as the negative of the distance:



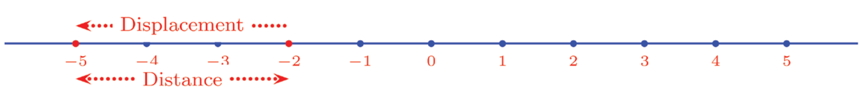
Displacement = 3
Distance = 3



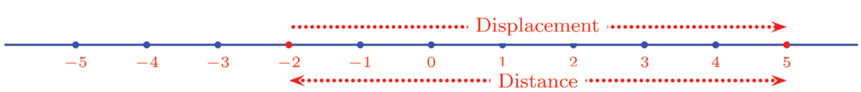
Displacement = −3
Distance = 3



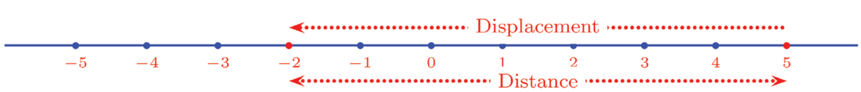
Displacement = 3
Distance = 3



Displacement = −3
Distance = 3



Displacement = 7
Distance = 7



Displacement = −7
Distance = 7

So, if the displacement is positive, then the distance is the displacement itself; if the displacement is negative, then the distance is the displacement with its negative sign removed. Thus distance is the absolute value of the displacement.

We have seen in the lesson, **Negative Numbers** that the displacement from the number x to the number y on the number line is $y - x$.

So, the distance between the numbers x and y is $|y - x|$

We have also seen that a number and its negative have the same absolute value; and the negative of $y - x$ is

$$-(y - x) = x - y$$

So, $|y - x| = |x - y|$

What do we see here?

The distance between two numbers x and y on the number line is $|x - y|$.

We can state this in another manner in ordinary language

If x is larger than y, then $x - y$ is positive so that $|x - y| = x - y$

Now if y is the larger number, then $x - y$ is negative, so that

$$|x - y| = -(x - y) = y - x$$

So what can we say in general?

The distance between two numbers on the number line is the number got on subtracting the smaller number from the larger number.

For example, the distance between $4\frac{1}{2}$, $6\frac{1}{4}$ is

$$6\frac{1}{4} - 4\frac{1}{2} = 1\frac{3}{4}$$

And the distance between $4\frac{1}{2}$ and $-6\frac{1}{4}$?

$$4\frac{1}{2} - \left(-6\frac{1}{4}\right) = 4\frac{1}{2} + 6\frac{1}{4} = 10\frac{3}{4}$$

This leads to another observation:

$|x - y|$ is the algebraic form of subtracting the smaller of the numbers x, y from the larger.

Now look at this problem:

What are the numbers x for which $|x - 1| = 3$?

We can do this in different ways

Geometrically, $|x - 1|$ is the distance between x and 1; and this distance is to be 3

The number to the right of 1, at a distance 3 is $1 + 3 = 4$

The number to the left of 1, at a distance 3 is $1 - 3 = -2$

Thus $x = 4$ or $x = -2$

Now let's think algebraically.

If $x > 1$, then $|x - 1| = x - 1$; and if $x - 1 = 3$, then $x = 4$

If $x < 1$, then $|x - 1| = 1 - x$; and if $1 - x = 3$, then $x = 1 - 3 = -2$

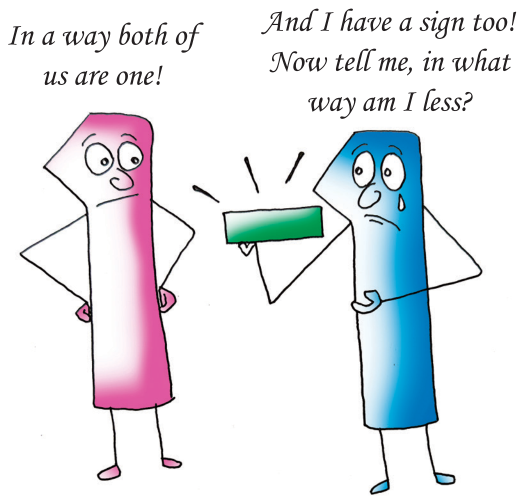
Let's change the problem slightly:

What are the numbers x for which $|x + 1| = 3$?

To do this geometrically, we must interpret $|x + 1|$ as a distance. Recall that distance is the absolute value of a difference. So, we must first rewrite $x + 1$ as a difference instead of a sum:

$$x + 1 = x - (-1)$$

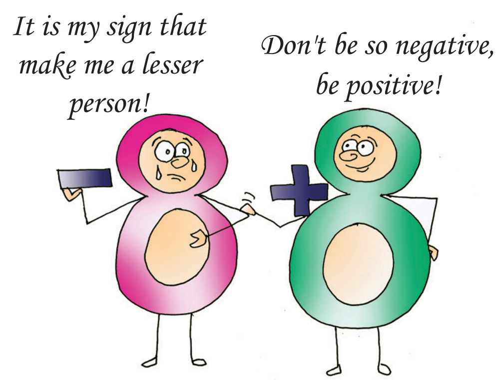
Using this, we can write $|x + 1|$ as $|x - (-1)|$ and hence as the distance between x and -1. Now as in the first problem, we can find the number to the right of -1, at a distance 3, as $-1 + 3 = 2$; and the number to the left of -1, at a distance 3, as $-1 - 3 = -4$.



Let's look at a different problem:

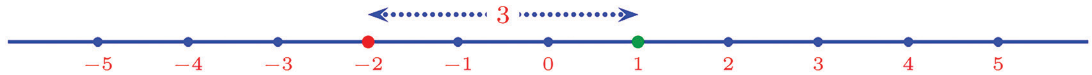
If $|x - 1| < 3$, then between what two numbers is x?

For this problem also, it is easier to reason geometrically:

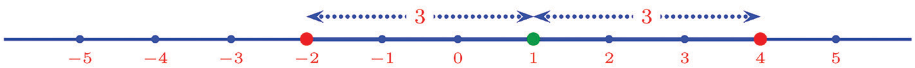


- The geometrical meaning of $|x - 1| < 3$ is that the distance between the numbers x and 1 is less than 3
- The number x can be either to the right of 1 or to left; only thing is, distance from 1 should be less than 3
    - ★ The number to the right of 1, at a distance 3 is $1 + 3 = 4$

★ The number to the left of 1, at a distance 3 is $1 - 3 = -2$



- If $|x - 1| < 3$, then the number should be between −2 and 4; that is $-2 < x < 4$



(1) Find x satisfying each of the equations below:

(i) $|x| = 5$ (ii) $|x - 3| = 2$

(iii) $|x - 2| = 3$ (iv) $|x + 2| = 3$

(2) Find between which numbers x should lie to satisfy each of the equations below:

(i) $|x| < 3$ (ii) $|x - 2| < 1$

(iii) $|x - 1| < 2$ (iv) $|x + 1| < 2$

(3) Find the integers satisfying each of the equations in problem (2)

Small and large

To satisfy $|x - 3| < 5$ the number x should be between −2 and 8 that is, $-2 < x < 8$

What about the x satisfying $|x - 3| > 5$?

If x is to the right of 3 at a distance farther than 5, or if x is to the left of 3 at a distance farther than 5, then this condition would be satisfied.

So, for this inequality to be satisfied, either x should be less than −2 or larger than 8; that is, $x < -2$ or $x > 8$.

Midpoint

On the number line, what is the number which marks the point exactly at the middle of the points marked by the numbers 2 and 4?

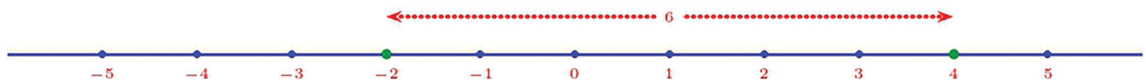
It is not difficult to see that this number is 3, is it? So, 3 can be geometrically described as the *midpoint* of 2 and 4

What is the midpoint of −2 and 4?

The distance from −2 to the midpoint should be half the distance from −2 to 4, right?

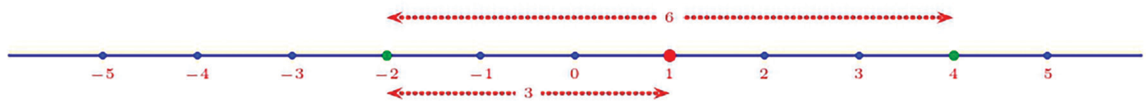
To get the distance from −2 to 4, the smaller number should be subtracted from the larger:

$$4 - (-2) = 6$$



Half this distance is 3. So, the number which marks the midpoint is at a distance 3 to the right of −2; that is

$$-2 + 3 = 1$$

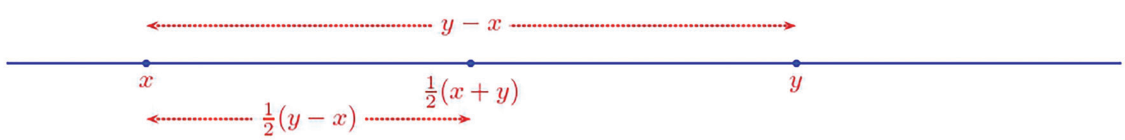


We can find the midpoint of any two numbers like this. Let's write this method using algebra. Let's denote the smaller of the numbers by x and larger by y. So on the number line, y is to the right of x at a distance $y - x$:

The midpoint is at half this distance to the right of x:

So, the midpoint is

$$x + \frac{1}{2}(y - x) = \frac{1}{2}x + \frac{1}{2}y = \frac{1}{2}(x + y)$$



The midpoint of two points on the number line is the point marked by half the sum of the numbers which mark those points

For example, the midpoint of $-2\frac{1}{2}$ and $4\frac{3}{4}$ is

$$\frac{1}{2} \times \left(-2\frac{1}{2} + 4\frac{3}{4}\right) = \frac{1}{2} \times 2\frac{1}{4} = 1\frac{1}{8}$$

and the midpoint of −4.8 and 1.2 is

$$= \frac{1}{2} \times (-4.8 + 1.2)$$

$$= \frac{1}{2} \times (-3.6) = -1.8$$

(1) Find the number which mark the midpoint of the points marked by each pair of numbers given below on the number line:

(i) 1, −5 (ii) −1, −5 (iii) $-\frac{1}{2}, -\frac{1}{3}$

(iv) $-\frac{1}{2}, \frac{3}{4}$ (v) −2.5, 3.5 (vi) 1.3, 8.7

(vii) $-\sqrt{2}, -\sqrt{3}$ (viii) $-\sqrt{3}, \sqrt{7}$

(2) Find the numbers which mark the points dividing the distance between the points marked by 1 and 2 into four equal parts, on the number line.

Number density

On any line, there are many points between any two points, however close they may be. For example, the midpoint of two points lie between them; the midpoints of this point and each of the first two points give two more points. And this can be continued indefinitely.

We can interpret this in terms of numbers. For any two numbers x and y, the number $z_1 = \frac{1}{2}(x+y)$ is between them. The numbers

$$z_2 = \frac{1}{2}(x+z_1), \quad z_3 = \frac{1}{2}(y+z_1),$$

are two other numbers between x and y. Thus we can find as many numbers as we want between x and y.

Now let's look at another problem:

What number x satisfies the equation $|x - 1| = |x - 4|$?

Think about the geometrical meaning of the equation

$$|x - 1| = |x - 4|$$

The point marked by x on the number line is at the same distance from the points marked by the numbers 1 and 4

What does this mean?

The point marked by x is the midpoint of the points marked by 1 and 4.

So,

$$x = \frac{1}{2} \times (1 + 4) = 2\frac{1}{2}$$

Find x satisfying each of the equations below:

(i) $|x - 1| = |x - 3|$ (ii) $|x - 3| = |x - 4|$

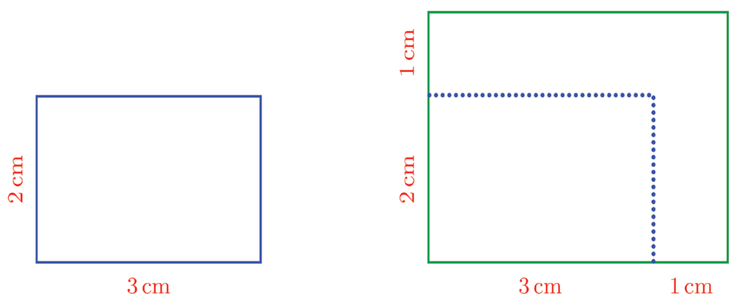
(iii) $|x + 2| = |x - 5|$ (iv) $|x| = |x + 1|$

10

POLYNOMIALS

Algebra of measurements

A rectangle of sides 2 centimetres and 3 centimetres is enlarged to a larger rectangle, by increasing each side by 1 centimetre:



What is the perimeter of the new rectangle?

The lengths of the sides of this rectangle are 3 centimetres and 4 centimetres so that perimeter is $2 \times (4 + 3) = 14$ centimetres.

We can think about this in another way. The perimeter of the original rectangle was $2 \times (3 + 2) = 10$ centimetres. Each side is increased by 1 centimetre, which means an increase of 4 centimetres in all. So the perimeter of the new rectangle is $10 + 4 = 14$ centimetres.

What if the sides are extended by 2 centimetres? If we use the second method, then the total increase in length is $4 \times 2 = 8$ centimetres, so that the new perimeter is $10 + 8 = 18$ centimetres.

This makes the computation easier, right? If the extension is $2\frac{1}{2}$ centimetres, then the new perimeter is

$$\left(4 \times 2\frac{1}{2}\right) + 10 = 20 \text{ cm}$$

In general, whatever be the extension of each sides, 4 times that added to 10 gives the new perimeter.

Let's write this using algebra. We denote the extension of each side as x centimetres and the new perimeter as p centimetres. Then we can write

$$p = (4 \times x) + 10 = 4x + 10$$

Now we can quickly compute the perimeters for various extensions:

If the extension is 3 centimetres, then the new perimeter is

$$(4 \times 3) + 10 = 22 \text{ centimetres}$$

If the extension is $3\frac{1}{2}$ centimetres, then the new perimeter is

$$\left(4 \times 3\frac{1}{2}\right) + 10 = 24 \text{ centimetres}$$

If the extension is $3\frac{3}{4}$ centimetres, then the new perimeter is

$$\left(4 \times 3\frac{3}{4}\right) + 10 = 25 \text{ centimetres}$$

Using algebra, these can be shortened:

$$\text{If } x = 3, \quad \text{then } p = 22$$

$$\text{If } x = 3\frac{1}{2}, \text{ then } p = 24$$

$$\text{If } x = 3\frac{3}{4}, \text{ then } p = 25$$

There is a way to shorten these still further:

$$p(3) = 22$$

$$p\left(3\frac{1}{2}\right) = 24$$

$$p\left(3\frac{3}{4}\right) = 25$$

In general, we can write

$$p(x) = 4x + 10$$

Let's look at this notation once again. First we write our computation in ordinary language:

> If the sides of a rectangle of sides two centimetres and three centimetres are all extended by the same length to make a larger rectangle, then the perimeter of this rectangle is ten added to four times the extension. For example if the sides are extended by $1\frac{1}{2}$ centimetres, then the perimeter becomes 16 centimetres.

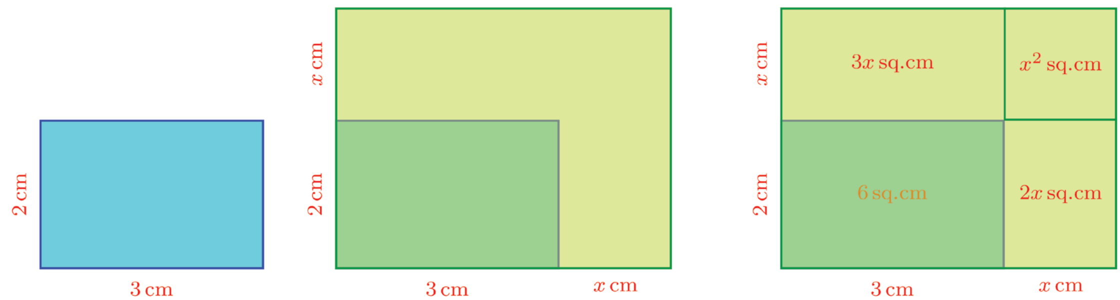
Next we write this in shorthand, using algebra:

> If the sides of a rectangle of sides 2 centimetres and 3 centimetres are all extended by x centimetres to make a larger rectangle, and the perimeter of the larger rectangle is p centimetres, then $p = 4x + 10$. For example, if $x = 1\frac{1}{2}$, then $p = 16$.

In this, p changes according to the change in x. To make this dependence clear, we write $p(x)$ instead of just p. Then we can change the above statement like this:

If the sides of a rectangle of sides 2 centimetres and 3 centimetres are all extended by x centimetres to make a larger rectangle, and the perimeter of the larger rectangle is $p(x)$ centimetres, then $p(x) = 4x + 10$. For example, $p\left(1\frac{1}{2}\right) = 16$

As another example, let's see how the area changes in the same setup. Instead of computing areas for various extensions as before, let's move straight to algebra, taking the extension of sides as x centimetres:



From the pictures above, we can see that the new area is

$$6 + 2x + 3x + x^2 = 6 + 5x + x^2$$

We can do this using only algebra without any pictures:

$$(3 + x)(2 + x) = 6 + 3x + 2x + x^2 = 6 + 5x + x^2$$

(The lesson, **Multiplication Identities**)

In algebraic expressions, we usually write the letters first. So, the above equation can be written

$$(x + 3)(x + 2) = x^2 + 5x + 6$$

As in the case of the perimeter problem, if the area got by extending each side by x centimetres is denoted $a(x)$ square centimetres, then we have

$$a(x) = x^2 + 5x + 6$$

Using this, we can compute for example

$$a(1) = 1^2 + (5 \times 1) + 6 = 1 + 5 + 6 = 12$$

$$a\left(1\frac{1}{2}\right) = \left(1\frac{1}{2}\right)^2 + \left(5 \times 1\frac{1}{2}\right) + 6 = 2\frac{1}{4} + 7\frac{1}{2} + 6 = 15\frac{3}{4}$$

$$a(2) = 2^2 + (5 \times 2) + 6 = 4 + 10 + 6 = 20$$

And these can be written in ordinary language like this:

If the extension is 1 centimetre, then the new area is 12 square centimetres

If the extension is $1\frac{1}{2}$ centimetres, then the new area is $15\frac{3}{4}$ square centimetres

If the extension is 2 centimetres, then the new area is 20 square centimetres

As another example, let's see how the volume of a rectangular block of edges 1 centimetre, 2 centimetres and 3 centimetres change when it is expanded to a larger block by extending all edges by the same length.

If the extension is denoted as x centimetres, then the volume of larger block is $(x + 1)(x + 2)(x + 3)$ cubic centimetres. To compute this product, we first write

$$(x + 2)(x + 3) = x^2 + 5x + 6$$

as seen earlier. Next this must be multiplied by $x + 1$. For that we must multiply each number in the first sum by each number in he second sum and add:

$$\begin{aligned}(x + 1)(x^2 + 5x + 6) &= (x \times (x^2 + 5x + 6)) + (1 \times (x^2 + 5x + 6)) \\ &= (x^3 + 5x^2 + 6x) + (x^2 + 5x + 6) = x^3 + 6x^2 + 11x + 6\end{aligned}$$

We write this in detail:

> If the sides of a rectangular block of edges 1 centimetre, 2 centimetres and 3 centimetres are all extended by x centimetres to make a larger rectangular block, and the volume of the larger block is $v(x)$ cubic centimetres, then $v(x) = x^3 + 6x^2 + 11x + 6$.

(1) In all rectangles with one side 1 centimetre less than the other, denote the length of the shorter side as x centimetres

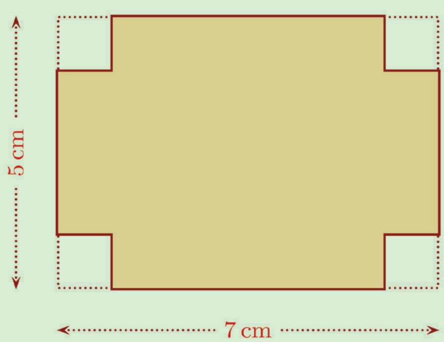
(i) Denote their perimeters as $p(x)$ centimetres and write the relation between x and $p(x)$ as an equation

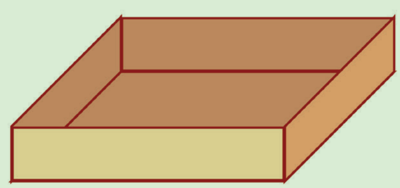
(ii) Denote their areas as $a(x)$ square centimetres and write the relation between x and $a(x)$ as an equation

(iii) Compute $p(1)$, $p(2)$, $p(3)$, $p(4)$, $p(5)$. Do you see any pattern?

(iv) Compute $a(1)$, $a(2)$, $a(3)$, $a(4)$, $a(5)$. Do you see any pattern?

(2) From the four corners of a rectangle, small squares of the same size are cut off and the tabs are raised up to make a box as in the picture below:





(i) Denote the length of the sides of the squares as x centimetres and write the the lengths of the three edges of the box in terms of x

(ii) Denote the volume of the box as $v(x)$ cubic centimetres and write the relation between x and $v(x)$ as an equation

(iii) Compute $v\left(\frac{1}{2}\right)$, $v(1)$ and $v\left(1\frac{1}{2}\right)$

(3) Consider all rectangles that can be made with a rope of length 1 metre. Denote the length of one side as x centimetres and the area enclosed by the rope as $a(x)$ square centimetres.

(i) Write the relation between x and $a(x)$ as an equation

(ii) Why are $a(10)$ and $a(40)$ the same number?

(iii) To get the same number as $a(x)$ when x is taken as two different numbers, what should be the relation between the numbers?

Special expressions

We have seen how some relations between measurements can be written as algebraic equations. These can be seen as relations between just numbers also. For example, in our first problem on rectangles, we wrote the relation between an extension of the sides and the new perimeter as

$$p(x) = 4x + 10$$

This can seen as the operation of multiplying a number by 4 and adding 10, beyond computation of perimeters

Let's have a look at the other relations we discussed earlier:

- $a(x) = x^2 + 5x + 6$
- $v(x) = x^3 + 6x^2 + 11x + 6$

If we look at them as operations on numbers, we can detect some common features. In all these, the only operations done are multiplying powers of the number x by fixed numbers and adding these products; and adding a fixed number. Algebraic expressions involving only these operations are called *polynomials*.

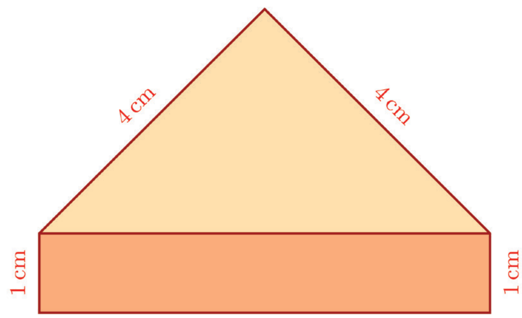
There are instances where operations other than these are done on measurements. For example, let's look at the computation of the diagonals of a rectangle with one side 1 centimetre longer than the other

If we denote the length of the shorter side as x centimetres, then the length of the diagonal in centimetres is

$$\sqrt{x^2 + (x+1)^2} = \sqrt{2x^2 + 2x + 1}$$

This involves the square root of varying numbers and so this is not a polynomial, by definition

Now look at this picture:



It shows a rectangle attached to the hypotenuse of an isosceles right triangle. What is its area?

The area of the triangle is easily seen to be 8 square centimetres. The length of the longer side of the rectangle is the hypotenuse of the isosceles right triangle and so is $4\sqrt{2}$ centimetres, as seen in the lesson **New Numbers**. The area of the rectangle is thus $4\sqrt{2}$ square centimeters.

The total area of the figure is $8 + 4\sqrt{2}$ square centimetres

What if the length of the perpendicular sides of the triangle is some other number? If we denote this number as x, then the area of the figure is the number

$$\frac{1}{2}x^2 + \sqrt{2}\,x$$

This uses the square root of 2; but the operations on the varying number x involves only squaring and multiplication by the fixed numbers $\frac{1}{2}$ and $\sqrt{2}$. So, this expression is indeed a polynomial.

Let's look at another example. Consider all rectangles of area 25 square centimetres. If we denote the length in centimetres of a side of such a rectangle as x, then the perimeter of the rectangle in centimetres is

$$2x + \frac{50}{x}$$

Since this expression involves the operation taking the reciprocal of the varying number x, it is not a polynomial.

A polynomial involves powers of a varying number. The largest exponent occurring like this is called the *degree* of the polynomial.

Look at some of the polynomials we have seen:

$$p(x) = 4x + 10$$

$$a(x) = x^2 + 5x + 6$$

$$v(x) = x^3 + 6x^2 + 11x + 6$$

In this list, the degree of the first polynomial is 1, the degree of the second is 2 and the degree of the third is 3

Instead of saying, "polynomial of first degree", "polynomial of second degree" and so on, we can simply say, "first degree polynomial", "second degree polynomial" and so on

We can write the general form of polynomials, based on their degrees:

First degree polynomial : $ax + b$

Second degree polynomial : $ax^2 + bx + c$

Third degree polynomial : $ax^3 + bx^2 + cx + d$

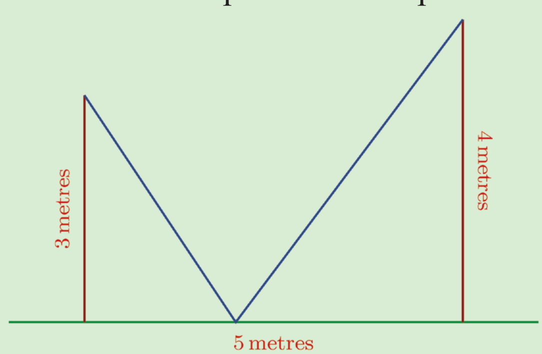
In these, a, b, c, d denote fixed numbers; thus when we take different numbers as x in such a polynomial, these numbers are not changed. They are called the *coefficients* of the corresponding powers. They can be any real numbers.

(1) In each of the problems below, check whether the relation between the specified measurements is a polynomial. Give reasons for your assertions.

(i) The relation between the length of the sides of a square park and the area of a 1 metre wide path around it.

(ii) The relation between the amount of acid added to a mixture of 3 litres of acid and 7 litres of water, and the change in the percent of acid in the mixture

(iii) Two poles of heights 3 metres and 4 metres stand 5 metres apart. A rope is to be stretched from the top of one post to some point on the ground and then stretched to the top of the other pole:



The distance from the foot of one pole to the point on the ground where the rope is fixed, and the total length of the rope.

(2) Write each of the following operations as an algebraic expression. Check which of them are polynomials, giving reasons

(i) Sum of a number and its reciprocal

(ii) Sum of a number and its square root

(iii) The product of the sum of a number and its square root, and the difference of the square root and the number

(3) For each of the polynomial $p(x)$ given below, compute $p(1)$ and $p(10)$

(i) $p(x) = 2x + 5$ (ii) $p(x) = 3x^2 + 6x + 1$ (iii) $p(x) = 4x^3 + 2x^2 + 3x + 7$

(4) For each of the polynomial $p(x)$ given below, compute $p(0)$, $p(1)$ and $p(-1)$

(i) $p(x) = 3x + 5$ (ii) $p(x) = 5x - 8$ (iii) $p(x) = 3x^2 + 6x + 1$

(iv) $p(x) = 2x^2 - 5x + 3$ (v) $p(x) = 4x^3 + 2x^2 + 3x + 7$

(vi) $p(x) = ax^3 + bx^2 + cx + d$

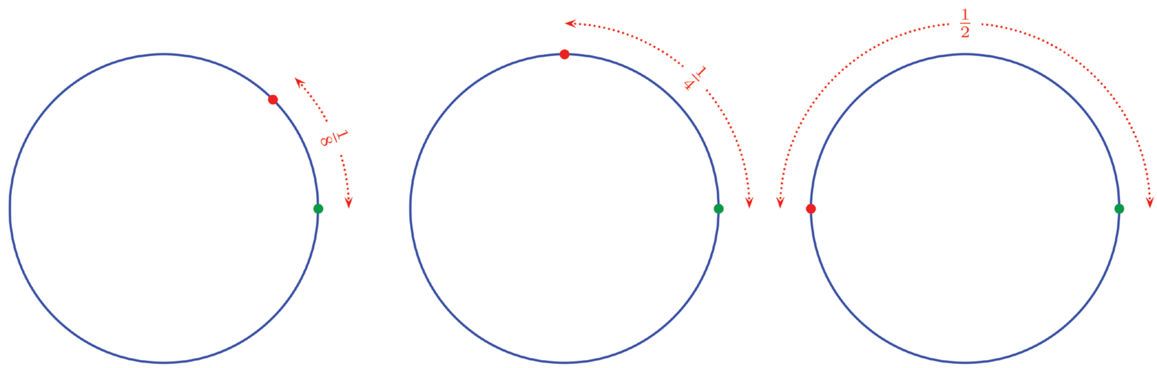
11

PARTS OF CIRCLES

Length and angle

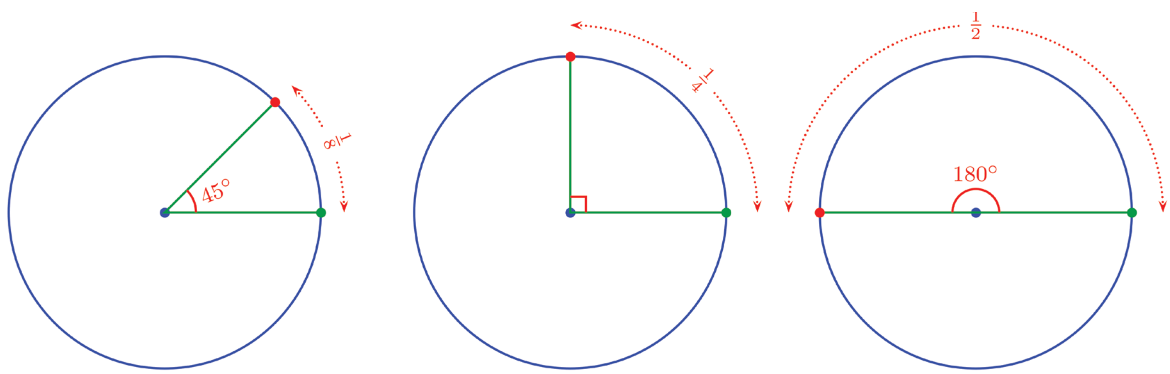
We have seen how we can calculate the perimeter and area of a circle. Now let's see how we can calculate the measures of some parts of a circle.

Consider a point travelling around a circle, starting from some point on the circle. The pictures below show the fraction of the circle covered at some stages of the journey:



Since the path of travel is a circle, instead of the *distance* travelled around the circle, we can also talk about the amount of *rotation* about the centre of the circle.

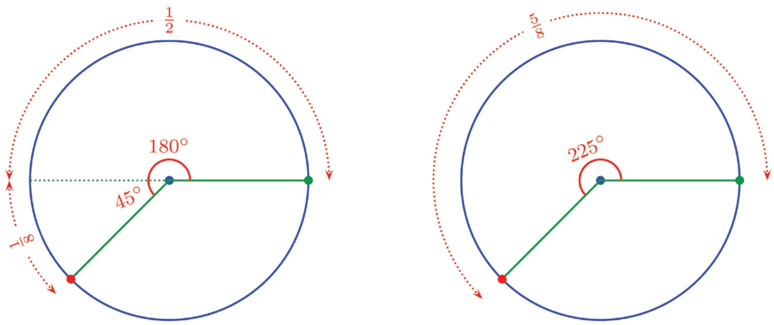
Remember how we drew angles at the centre of a circle to get different parts of a circle, in the lesson **Angles** in class 6? To get $\frac{1}{8}$ of a circle, an angle of $360° \div 8 = 45°$ and to get $\frac{1}{4}$ of a circle, an angle of $360° \div 4 = 90°$ at the centre and so on.



Thus we can describe the journey either in terms of lengths or angles.

Now a question: after completing half a rotation and moving one eighth of the circle more, the distance travelled is $\frac{1}{2} + \frac{1}{8} = \frac{5}{8}$ of the circle. How do we describe this as a rotation?

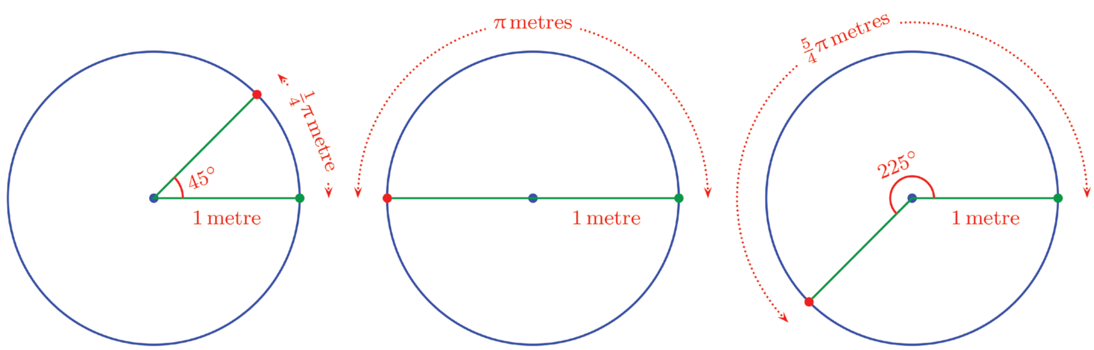
$\frac{1}{8}$ of the circle means an angle of 45°; so after a rotation of 180°, there is a further rotation of 45°. So, we can say a total rotation of 180° + 45° = 225°:



Thus each stage of the complete journey around the circle, till arriving back at the starting position, can be described either as a fraction of the circle or as the amount of rotation in degrees (up to 360°).

In GeoGebra draw a circle centred at a point A, passing through the point B. Create a slider a, choosing the option Angle. Select the Angle with Given Size and click on B and then A. Give the angle asked for as a. We get a new point B' on the circle. Dragging the slider moves B' around the circle

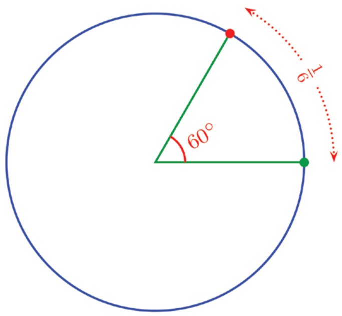
Now suppose we take the radius of the circle as 1 metre. Then the circumference of the circle is 2π metres; and we can describe the distance travelled as actual lengths, instead of fractions of the circle:

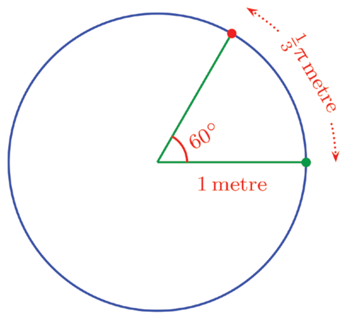


So we can describe each stage of the journey either in terms of the distance travelled in metres, or in terms of the the amount of rotation in degrees

If the rotation is 60°, what is the distance travelled around the circle?

First let's calculate it as a fraction of the whole circle. 1° is $\frac{1}{360}$ of the circle, right? So, 60° is $60 \times \frac{1}{360} = \frac{1}{6}$ of the circle. Since the perimeter of the circle is 2π metres, this is $\frac{1}{6} \times 2\pi = \frac{1}{3}\pi$ metres.

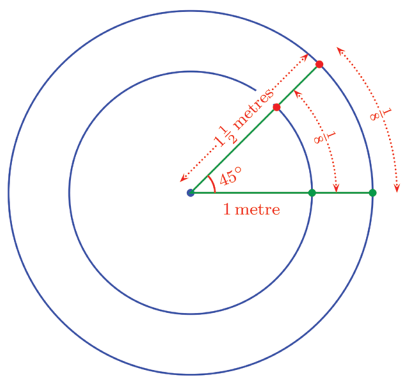


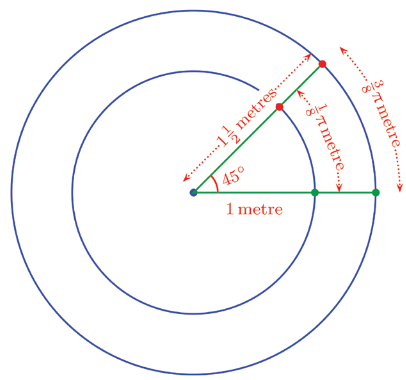


In general. whatever fraction of 360° is the rotation, that fraction of 2π metres is the distance travelled

What if the radius of the circle is $1\frac{1}{2}$ metres? Then the circumference is 3π metres. So to compute the distance corresponding to a rotation, we must take fractions of 3π. In other words, though the fraction of circumference corresponding to a rotation does not change; but the actual distance in metres changes.

For example, if rotation is 45°, the distance is still $\frac{1}{8}$ of this circle; but since the circle is larger, the actual distance changes to $\frac{3}{8}\pi$ metres:





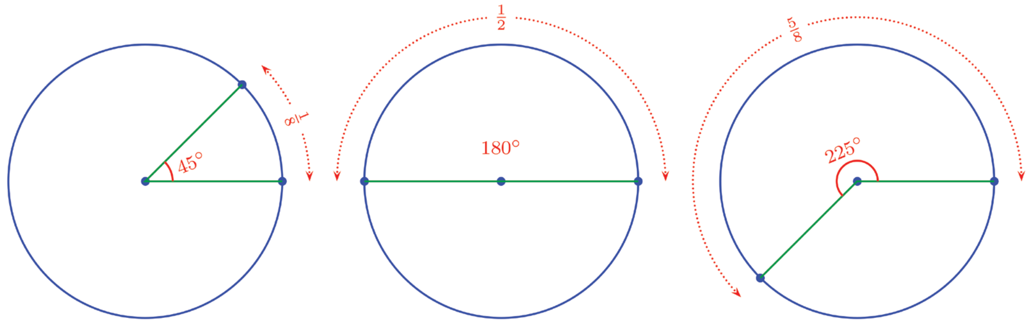
This is the general result:

> In a circle of radius r, the distance travelled around the circle for a rotation of x° about the centre is $\frac{x}{360} \times 2\pi r$

Now let's see how this is stated using some new mathematical terms. The part of a circle between two points on the circle is called an *arc* of the circle. The angle of rotation from one end of the arc to the other end is called the *central angle* of the arc.

Draw circle with centre at A in GeoGebra and mark two points B and C on it. Hide the circle by right clicking on it and unchecking the Show Object in the menu. Select the Circular Arc tool and click first on A and then on B and C to get an arc. Clicking on A and then on C and B results in another arc. See the difference. Drawing the radii AB, AC, we can get the angle between them using the Angle tool

So, according to what we have seen before, we can say that for an arc which is $\frac{1}{8}$ of the whole circle, the central angle is 45°, for an arc which is half the circle, the central angle is 180°, for an arc which is $\frac{5}{8}$ of the circle, the central angle is 225° and so on.



The relation between distance travelled around a circle and the angle of rotation about the centre can now be stated as a result in pure mathematics.

In a circle of radius r, if the central angle is x°, then arc length is $\frac{x}{360} \times 2\pi r$

> Whatever fraction of 360° is the central angle of an arc, that fraction of the circumference of the circle is the length of the arc

For example, in a circle of radius 3 centimetres, what is the length of an arc of central angle 60°?

We can do the computation mentally: since 60° is $\frac{1}{6}$ of 360°, the length of the arc is $\frac{1}{6}$ of the circumference of the circle, which is 6π centimetres; so the length of the arc is π centimetres.

What about the length of an arc of central angle 50° in a circle of radius 2.5 centimetres?

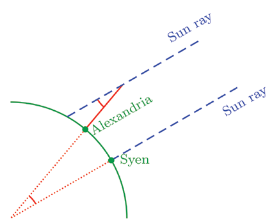
The length of the arc is $\frac{50}{360}$ of the circumference, and the circumference is 5π centimetres. So, the length of the arc in centimetres is

$$\frac{50}{360} \times 5\pi = \frac{25}{36}\pi \approx 2.2 \text{ centimetres.}$$

Circumference of the earth

The circumference of the earth was first calculated by the Greek scholar Eratosthenes in the second century BCE

He knew on a particular day of each year, the midday sun in the city of Syene would be directly overhead, so that no shadows would be cast. He measured the slant of the sun's rays at that time in his native city of Alexandria, using the shadow of a vertically erected stick:



If we assume that the rays of the sun reach earth as parallel lines, then this angle is equal to the central angle of the arc joining Syene and Alexandria. And the distance between the two cities is the length of the arc

Denoting the slant of sun's rays at Alexandria as a° and the distance to Syene as d, the circumference of the earth can be calculated as $\frac{360}{a} \times d$

Let's look at another problem:

From an iron ring of radius 9 centimetres, a piece of central angle 30° was cut off and this piece was bent into a small circle. What is its radius?

The length of an arc of central angle 30° is $\frac{1}{12}$ of the circumference. So, the length of the piece is $\frac{1}{12} \times 18\pi = \frac{3}{2}\pi$ centimetres. This is the circumference of the small circle. So its radius is $\frac{3}{2}\pi \div 2\pi = \frac{3}{4}$ centimetre.

Make the GeoGebra applet of the previous experiment, with the radius of the circle $\frac{12}{\pi}$ (Give the radius as 12/pi). What is the circumference of this circle? Mark the length of the arc using the Distance or Length tool. What is arc length for central angle 60°? How do we draw an arc of length 2? Try drawing arcs of lengths 3, 5, 15, 18 and 19

There's an easier way to compute this: the circumference of the small circle is $\frac{1}{12}$ of the circumference of the large circle. Since the radius and circumference are scaled by the same factor, the radius of the small circle is $\frac{1}{12}$ of the radius of the large circle; that is, $\frac{1}{12}$ times 9.

$$\frac{1}{12} \times 9 = \frac{3}{4} \text{ centimetre}$$

(1) In a circle, the length of an arc of central angle 40° is 3π centimetres. What is the circumference of the circle? And the radius?

(2) In a circle, the length of an arc of central angle 25° is 4 centimetres.

(i) What is the length of an arc of central angle 75° in this circle?

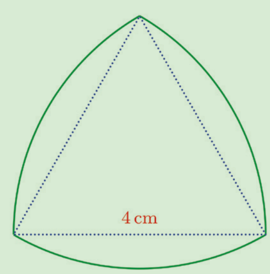
(ii) What is the length of an arc of central angle 75° in a circle with radius one and a half times this circle?

(3) From a bangle of radius 3 centimetres, a small piece is to be cut off to make a ring of radius $\frac{1}{2}$ centimetre

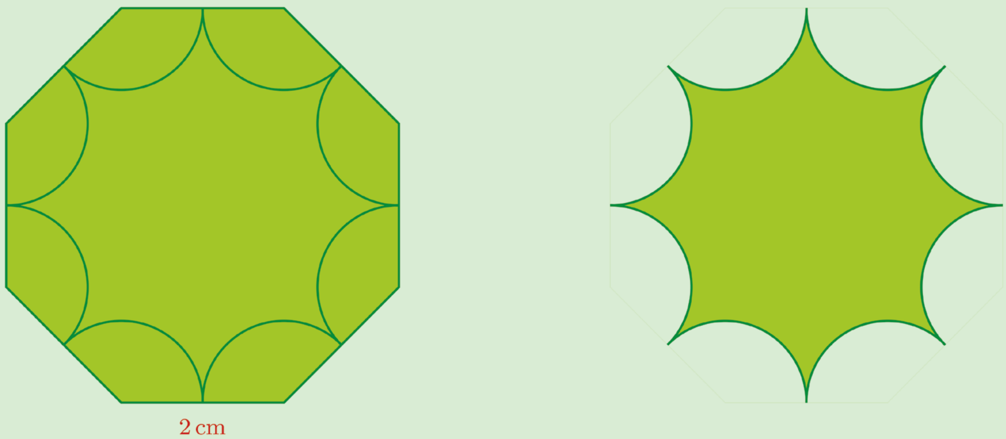
(i) What should be the central angle of the piece to be cut off?

(ii) The remaining part of the bangle is bent to make a smaller bangle. What is its radius?

(4) With each vertex of an equilateral triangle as center, an arc of a circle which passes through the other two vertices is drawn to get the given figure. What is its perimeter?



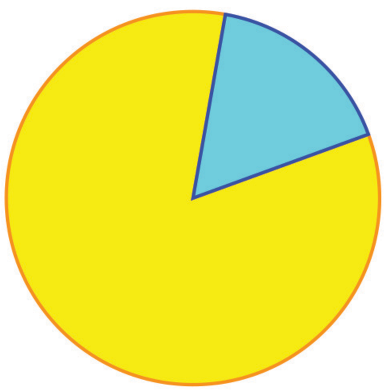
(5) With each vertex of a regular octagon as center an arc of a circle is drawn and the resulting figure is cut off, as in the pictures below:



What is the perimeter of the figure cut off?

Angles and areas

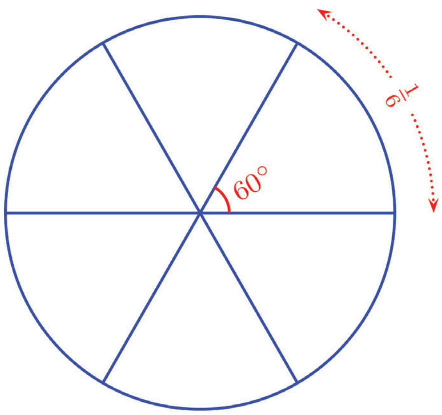
An arc is a part of the circumference of a circle. Combined with the radii through its ends, it forms the region of the circular area:

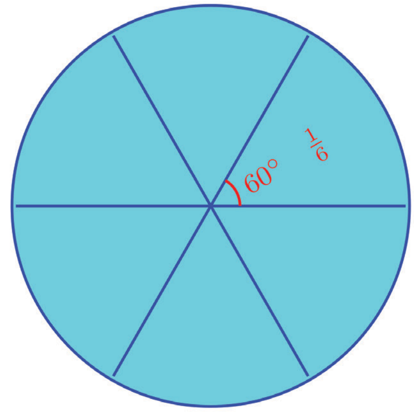


In GeoGebra, draw a circle with center A and mark two pints B and C on it. Selecting the Circular Sector tool and clicking on A, B, C in this order, we can draw a sector of the circle. Selecting the Area tool and clicking inside the sector, we get its area

Such a part is called a *sector* of the circle. The central angle of the arc of a sector is called the central angle of the sector also.

Just as the length of an arc changes with the central angle, so does the area of the sector; and the computations are similar. For example, an arc of central angle $60°$ is $\frac{1}{6}$ of the circumference; and the area of a sector of central angle $60°$ is $\frac{1}{6}$ of the area of the circle.





Like this, an arc of central angle 1° is $\frac{1}{360}$ of the circumference of the circle and a sector of central angle 1° is $\frac{1}{360}$ of the area of the circle. Thus the relation between the central angle and the area of a sector can be stated in the same way as the relation between the central angle and length of an arc:

> Whatever fraction of 360° is the central angle of a sector, that fraction of the area of the circle is the area of the sector

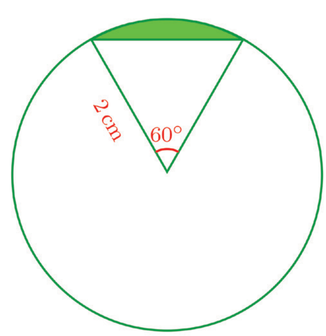
> Draw a circle in GeoGebra with center at A and radius $\sqrt{\frac{240}{\pi}}$. This can be done by using the tool Circle: Centre & Radius and giving the radius as sqrt(240/pi). What is the area of this circle? Mark a point B on the circle and as before, a moving point B' on the circle, using an angle slider. Draw the sector joining A, B, B' using the Sector tool and mark its area. What is the area of the sector with central angle 60°? Draw sectors of area 20, 44, 100, 120, 194. Find their central angles

Using algebra, this can be stated like this:

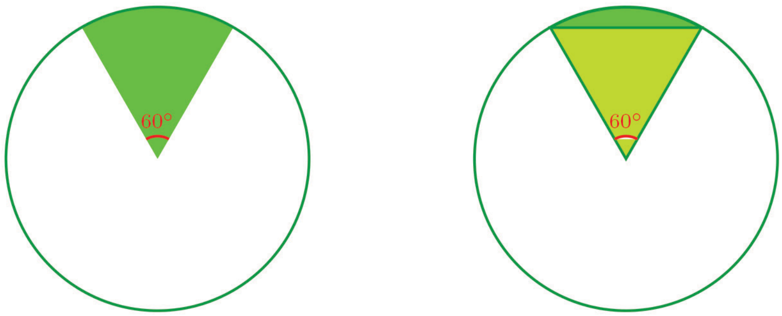
The area of a sector of central angle $x°$ in a circle of radius r is $\frac{x}{360} \times \pi r^2$

For example, the area of a sector of central angle 40° is $\frac{40}{360} = \frac{1}{9}$ of the area of the circle. If the radius of the circle is 3 centimetres, its area is 9π square centimetres and so the area of this sector is π square centimetres.

Now a problem: What is the area of the green region of the circle in this picture?



We get this region by removing a triangle from a sector:



The area of the sector is $\frac{1}{6}$ of the area of the circle

That is, $\frac{1}{6} \times 4\pi = \frac{2}{3}\pi$ square centimetres.

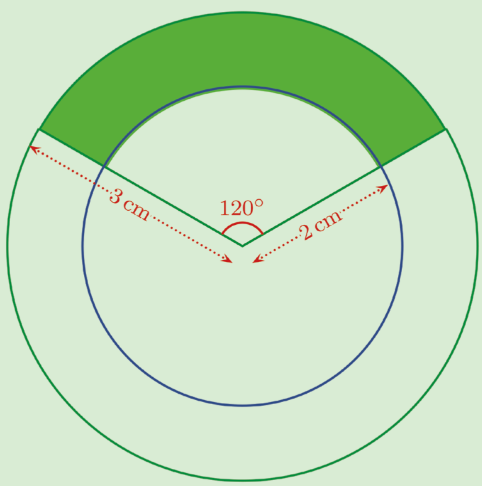
Since the triangle is equilateral (why?) its area is $\sqrt{3}$ square centimetres.

So the area of the green region in the first picture is $\frac{2}{3}\pi - \sqrt{3}$ square centimetres.

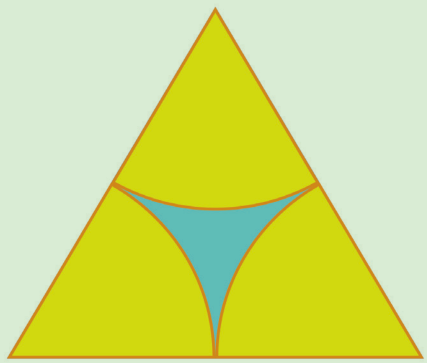
(1) (i) In a circle of radius 3 centimetres, What is the area of a sector of central angle 120°?

(ii) What is the area of a sector of the same central angle in a circle of radius 6 centimetres?

(2) Find the area of green region in the picture below:



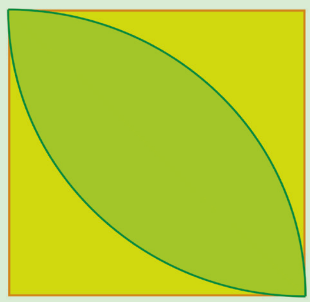
(3) In the picture below, arc of a circle is drawn with each vertex of an equilateral triangle as centre and half the side as radius:



4 cm

Calculate the area of the blue region.

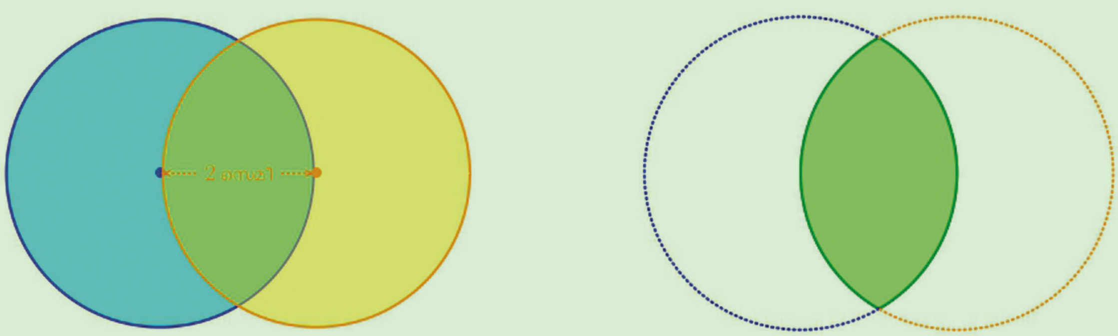
(4) In the picture below, arcs of circles are drawn centred on two opposite vertices of a square and passing through the other two vertices:



4 cm

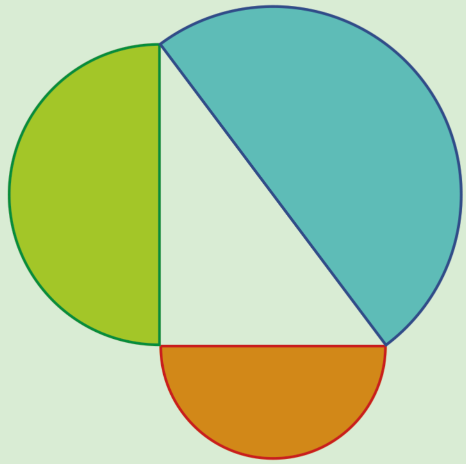
What is the area of green region?

(5) The picture below shows two circles of the same radii, each passing through the centre of the other:

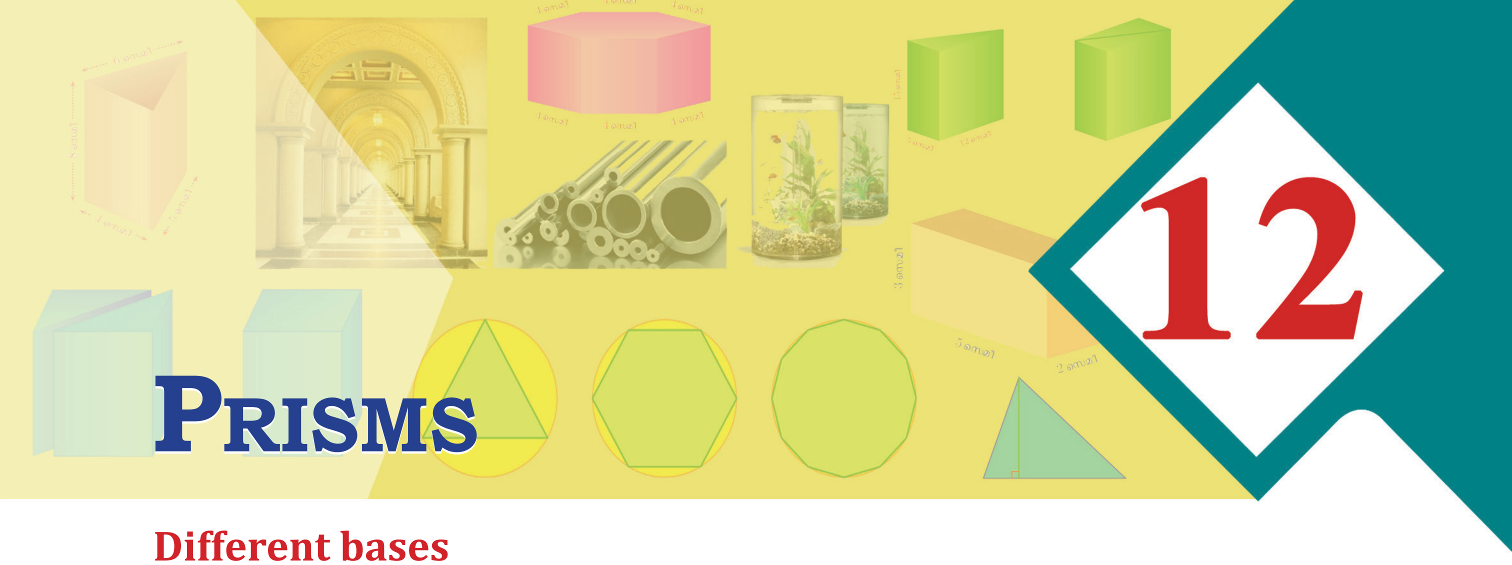


Calculate the area of the region common to both.

(6) The picture below shows semicircles drawn with the sides of a right triangle as diameters:



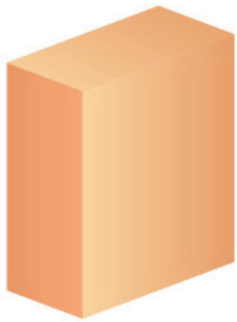
Prove that the area of the largest semicircle is the sum of the areas of the other two.



12 PRISMS

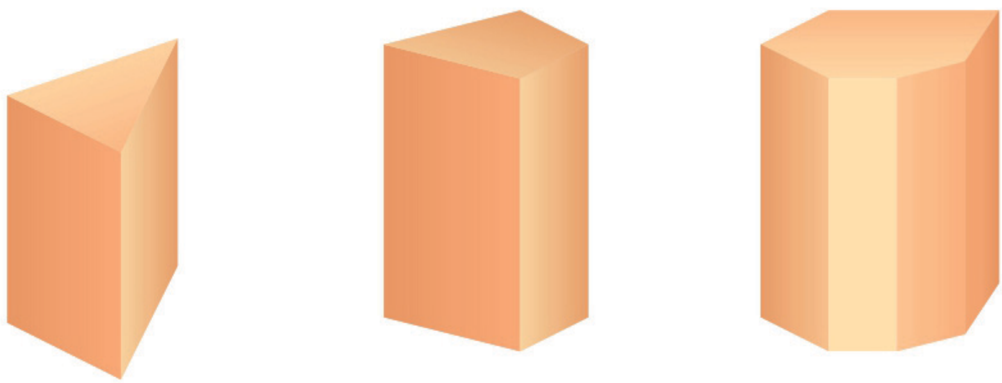
Different bases

You have learned about rectangular blocks and their volumes in class 6, haven't you?



Its surface is made up of six rectangles; pairs of the same size at the top and bottom, left and right, front and back; six altogether.

See these pictures:



All these shapes, with horizontal spread and vertical height, are said to be *three-dimensional objects* or *solids.*

The three-dimensional objects shown here have some other peculiarities.

What can we say about their surfaces?

In the first, equal triangles at the top and bottom and three rectangles on the sides; in the second, quadrilaterals on top and bottom and four rectangles on the sides, and in the third, hexagons at the top and bottom and six rectangles on the sides

In general, the surface of each of them is made up of two identical polygons on the top and bottom, and rectangles of the same height with corresponding sides of the polygons as a pair of opposite sides. Such solids are called *prisms*.

The top and bottom polygons and the rectangles on the sides are called *faces* of the prism; those faces on the top and bottom are called *bases* and those on the sides are called *lateral faces*. The sides of the faces are called *edges*.

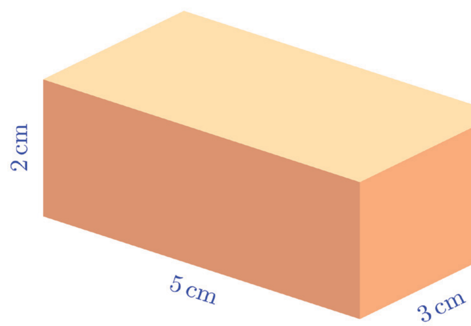
Prisms can be classified as triangular, quadrangular, hexagonal and so on according to the base. The pictures above show a triangular prism, quadrangular prism and a hexagonal prism. What we have called a rectangular block so far can now be called (in formal language) a rectangular prism

Let's see how we can draw prisms in GeoGebra. First we draw the base, using the Polygon tool. Next open the 3D graphics window via View → 3D Graphics. The polygon drawn will be visible in this window also. Clicking on this window shows a new set of tools in the toolbar. Choose the tool Extrude to Prism from these and click on the polygon in the 3D window. This opens up a window in which the height (Altitude) of the prism can be given. This draws the prism. To remove the labels of vertices and edges, first select Options→Labeling→No New Objects, before starting to draw

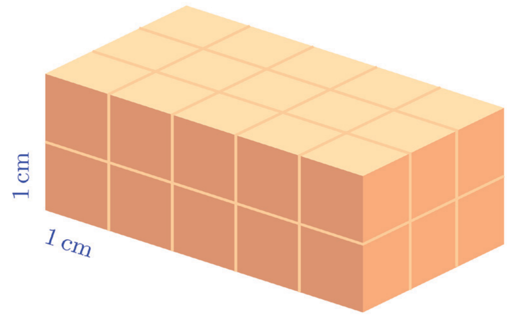
Try making hollow prisms by cutting out cardboard polygons and rectangles

Volume

Remember how we calculated volumes of rectangular prisms (blocks) in class 6? For example, look at this rectangular prism:



This can be split into cubes of edges 1 centimetre:

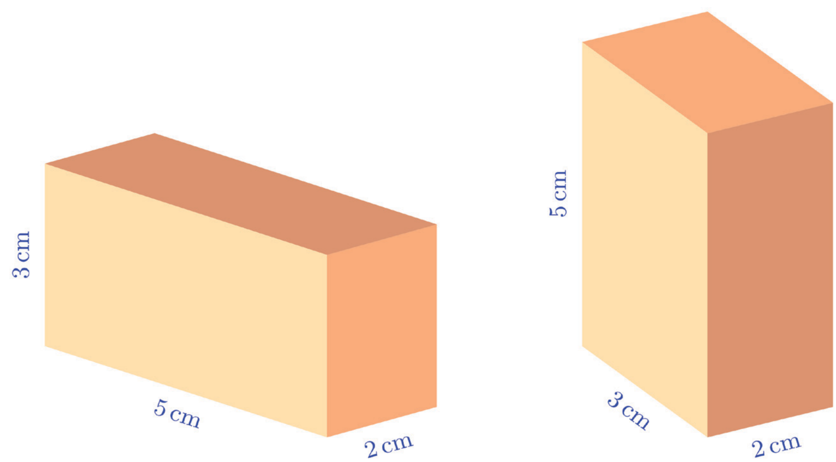


It contains 5 × 3 × 2 = 30 cubes; so its volume is 30 cubic centimetres.

In class 7, we have seen that even if the lengths of the sides of a rectangle are fractions, its area is their product (the section, **Fractional area** of the lesson, **Fractions**). In the same way we can show that the volume of a rectangular prism is the product of their edges, even when they are fractions. Again, in class 9 we have seen that even if the length of sides of a rectangle are irrational numbers, the area is still their product (the section **Multiplication** of the lesson **Irrational Multiplication**). This can also be extended to get the volume of a rectangular prism with lengths of sides irrational as the product of these lengths.

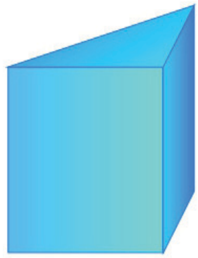
The volume of a rectangular prism can be described in a different manner. In the prism shown above, the base is a rectangle of sides 5 centimetres and 3 centimetres and hence area 5 × 3 square centimetres. So, the volume is the product of this area and its height, 2 centimetres

Since all faces of a rectangular prism are rectangles, we can take any of these as the base:

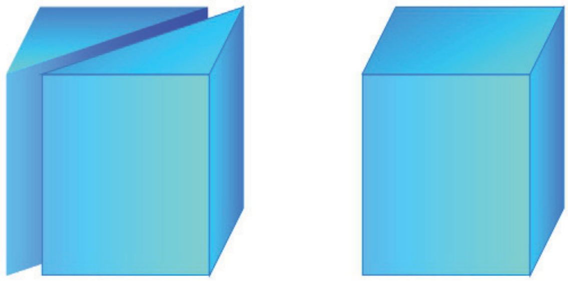


Whichever base we choose, volume is the product of its base area and the corresponding height, isn't it?

Let's see if we can compute the volume of any prism like this. First we take a right triangular prism:

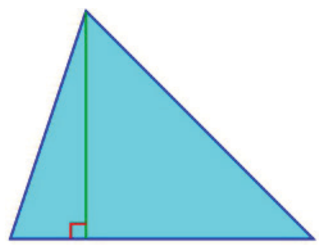


We can make a rectangular prism by attaching another identical prism to it:

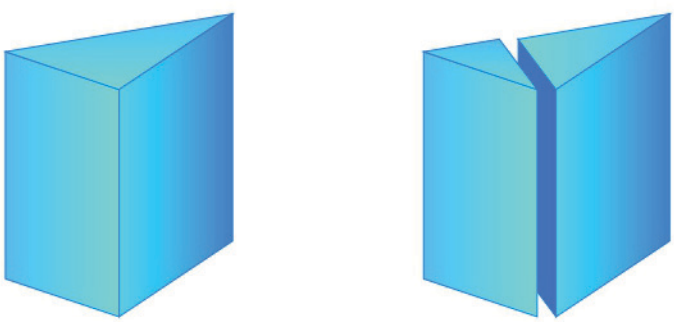


If we denote the base area of the right triangular prism by a, the base area of the rectangular prism got by attaching its duplicate is $2a$. The heights of the triangular prism and the rectangular prism are the same. Denoting it by h, the volume of the rectangular prism is $2ah$. Since this is double the volume of two of the triangular prisms, the volume of a single triangular prism is ah; that is its base area multiplied by height.

What about a triangular prism whose base is not right angled? We know how any triangle can be split into two right triangles by drawing the perpendicular from a vertex to the opposite side:



So, any triangular prism can be split into two right triangular prisms by slicing vertically through such a perpendicular in the top face, as shown below:

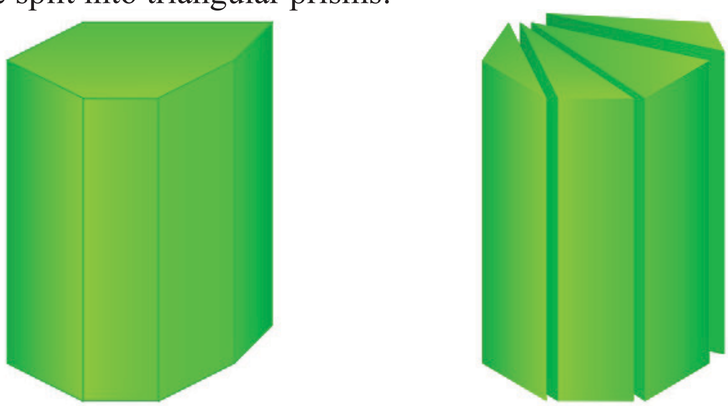


Adding the volumes of these right triangular prisms will give the volume of the original triangular prism. If we denote the base area of the original triangular prism by a and the base areas of the two right triangular prisms by b and c, then $a = b + c$. All three prisms have the same height. Denoting this by h, the sum of the volumes of the right triangular prisms is $bh + ch = (b + c)h = ah$, which is the product of the base area and height of the original triangular prism.

Thus the volume of any triangular prism is the product of its base area and height.

Now we can divide any polygon into triangles by joining one vertex to all others:

And the area of the polygon is the sum of the areas of these triangles. So any prism can be split into triangular prisms:



In GeoGebra draw a polygon using the Polygon tool. (The number of vertices can be six or seven.) Its area can be got using the Area tool. Draw a prism of any height you like, with this polygon as base. Selecting the Volume tool and clicking inside the prism gives its volume. What is the relation between base area, height and the volume? Check by changing the positions of the vertices. (By dragging a vertex onto another, the number of vertices can be reduced)

And the base area is the sum of the base areas of the pieces. Let's denote the base area of a whole prism by a and its height by h. If the base of the prism is split into n triangles, the prism itself is split into n triangular prisms. Let's denote their base areas by $b_1, b_2, ..., b_n$. Then their volumes are $b_1h, b_2h, ..., b_nh$. So, the volume of the original prism is

$$b_1h + b_2h + \ldots + b_nh = (b_1 + b_2 + \ldots + b_n)h = ah$$

Thus we have a general result:

The volume of any prism is the product of the base area and the height.

For example, consider this problem:

The base of a prism is an equilateral triangle of sides 4 centimetres and its height is 10 centimetres. What is its volume?

We have seen that the area of an equilateral triangle is $\sqrt{3}$ times the square of half the side, in the lesson, **Irrational Multiplication**. In this problem, the base is an equilateral triangle of side 4 centimetres. So, the base area in centimetres is

$$\sqrt{3} \times 2^2 = 4\sqrt{3}$$

Since the height is 10 centimetres, the volume is

$$4\sqrt{3} \times 10 = 40\sqrt{3} \text{ cubic centimetres}$$

Do these problems using GeoGebra. Draw a prism with base an equilateral triangle of perimeter 15 centimeters, and height 5 (The Regular Polygon tool can be used to draw an equilateral triangle). What is the volume of this prism?

All the edges of a triangular prism are equal and its volume is $16\sqrt{3}$. What is the length of its edges? Draw this prism. The volume of a hexagonal prism with all edges equal is $12\sqrt{3}$. Draw this prism.

(1) The base of a prism is an equilateral triangle of perimeter 18 centimetres and its height is 5 centimetres. Calculate its volume.

(2) The base of a prism is a triangle of sides 13 centimetres, 14 centimetres and 15 centimetres, and its height is 20 centimetres. Calculate its volume.

(3) There is a hexagonal pit in the school ground to collect rain water. Each side of the hexagon is 2 metres and the depth of the pit is 3 metres. It now contains water one metre deep. How much litres is this? How much more litres can it contain?

(4) A hollow prism with base a square of sides 16 centimetres contains water 10 centimetres high. If a cube of edges 8 centimetres is immersed in it, by how much would the water level rise?

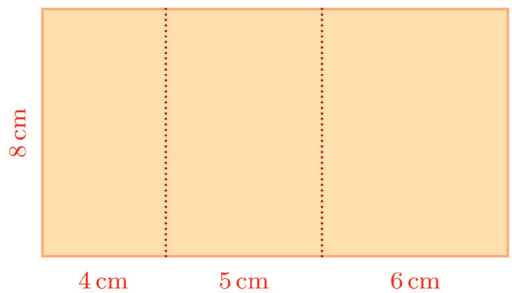
(5) A rectangular block of metal has edges of lengths 6 centimetres, 9 centimetres and 15 centimetres. It is melted and recast into identical cubes of sides 3 centimetres. How many cubes would be got?

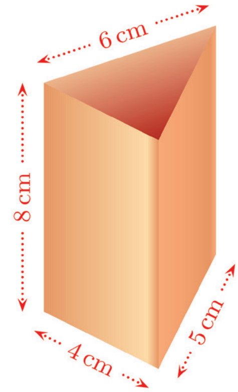
(6) The base of a prism is a square of sides 6 centimetres and its height is 10 centimetres. What is its volume? What is the maximum volume of a triangular prism cut from it?

Area

A hollow tube is to be made of thick paper, with dimensions as shown in the picture

It can be made by gluing together three rectangles cut out separately or by folding a single rectangle like this:





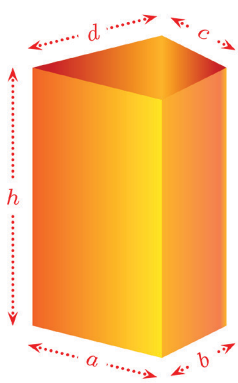
What is the area of the paper used to make this prism? It is the area of the rectangle above, which is in square centimetres

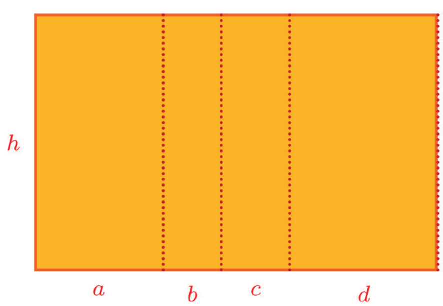
$$(4 + 5 + 6) \times 8 = 15 \times 8 = 120 \text{ square centimetres}$$

It is the sum of the areas of the lateral faces of the prism, isn't it? In general, the sum of the areas of lateral faces of a prism is called its *lateral surface area.*

We computed the lateral surface area of the triangular prism above by multiplying 15 by 8. In this 15 = 4 + 5 + 6 is the perimeter of the triangle which forms the base of the prism and 8 is the height of the prism. So, the lateral surface area of this prism is the product of the base perimeter and height. A little thought convinces us that this can be done for any triangular prism.

What if the base is a quadrilateral instead of a triangle?





Imagine a hollow prism cut along some of the edges and spread flat as a single sheet. It can be folded back into the prism also. To do this for a prism in GeoGebra, select the Net tool and click on the prism. Draw some prisms and see what their nets are.

The lateral surface area of this prism is the area of the rectangle got by slicing and spreading the prism, isn't it? And that is $(a + b + c + d)h$, which is again the product of the base perimeter and height.

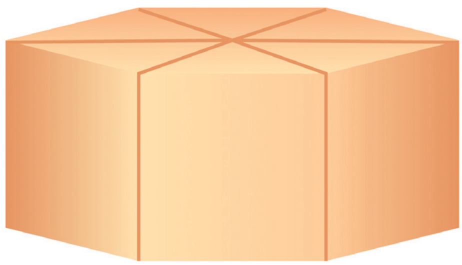
We can compute the lateral surface area of any prism like this:

The lateral surface area of any prism is the product of its base perimeter and height.

For a closed prism, the total surface area can be calculated by just adding twice the base area to the lateral surface area.

Let's look at a problem:

The base of a wooden prism is an equilateral triangle. Its lateral surface area is 48 square centimetres and its height is 4 centimetres. Six of these are joined to make a hexagonal prism:



It is to be prettied up by sticking coloured paper on all its faces. What is the area of paper needed?

What we need here is the total surface area of the hexagonal prism; and for that we have to add the lateral surface area and twice the base area.

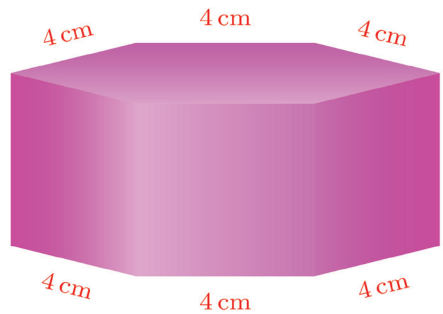
To calculate the lateral surface area, we need the perimeter of the hexagon and for that we need the length of the sides of the base of the triangular prisms.

We can find the base perimeter of any prism by dividing the lateral surface area by the height, right?

So the base perimeter of the triangular prism is $48 \div 4 = 12$ centimetres.

Since the base is an equilateral triangle, its perimeter is three times the length of a side; so the length of a side of the triangle in our problem is $12 \div 3 = 4$ centimetres

Now we can calculate the base perimeter of the hexagonal prism, can't we?



The perimeter of a hexagon with all sides 4 centimetres is $6 \times 4 = 24$ centimetres. Since the height of the prism is also 4 centimetres, its lateral surface area is $24 \times 4 = 96$ square centimetres.

To this we must add the areas of both the bases. The area of each of the six triangles which make the hexagon is

$$\sqrt{3} \times 2^2 = 4\sqrt{3} \text{ square centimetres.}$$

The area of the hexagon formed by six of them is $6 \times 4\sqrt{3} = 24\sqrt{3}$ square centimetres.

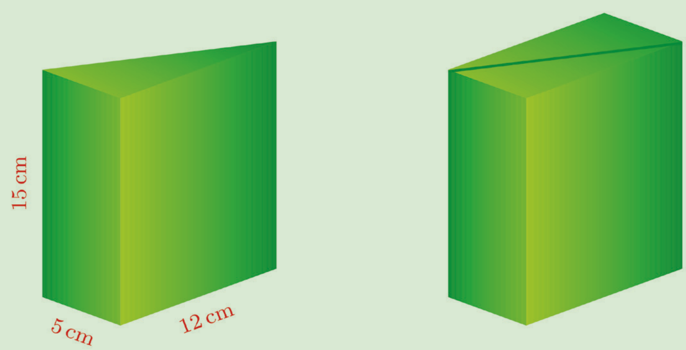
So, the total surface area of the hexagonal prism is

$$96 + (2 \times 24\sqrt{3}) = 96 + 48\sqrt{3} = 48(2 + \sqrt{3}) \text{ square centimetres}$$

We can use a calculator to see that this number is a little more than 179. Anyhow, 180 square centimetres of paper would be enough.

(1) The base of a prism is an equilateral triangle of perimeter 12 centimetres and its height is 5 centimetres. What is its total surface area?

(2) Two identical prisms with right triangles as bases are joined to form a rectangular prism, as in the picture below:



What is the total surface area of this rectangular prism?

(3) The base of a prism is a triangle with sides 4 centimetres, 13 centimetres and 15 centimetres and its height is 25 centimetres. Calculate its lateral surface area and total surface area.

(4) The lateral surface area of a prism, with base an equilateral triangle, is 120 square centimetres

(i) What is the lateral surface area of a prism, with base a rhombus, made by joining two such triangular prisms?

(ii) What is the lateral surface area of a prism, with base an isosceles trapezium, made by joining three such triangular prisms?

(iii) What is the lateral surface area of a prism, with base a regular hexagon, made by joining six such triangular prisms?

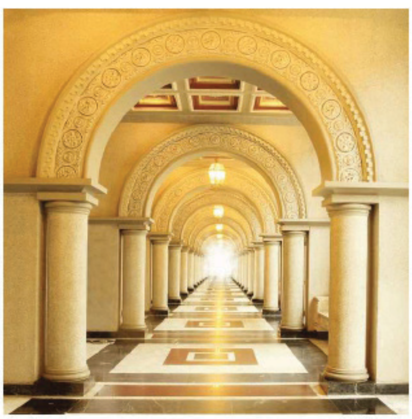
(5) Six sheets of metal, each a square of sides 10 centimetres are joined to make a cube

(i) What is its total surface area?

(ii) How much water can it contain?

Cylinders

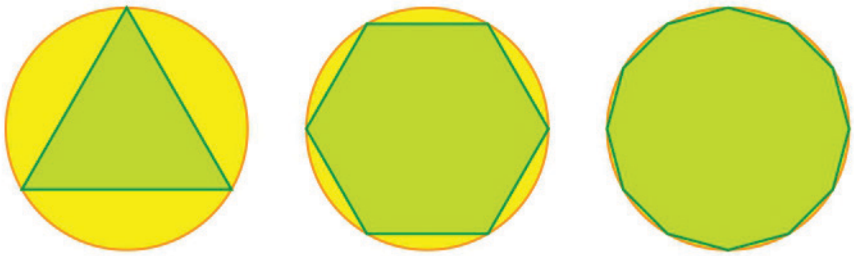
Prisms are three-dimensional objects with polygons at the ends and rectangles on the sides. There are also three-dimensional objects with circles at the ends and the side smoothly rolled curves, instead of being bent into rectangles. You might have seen several such objects, both solid and hollow:



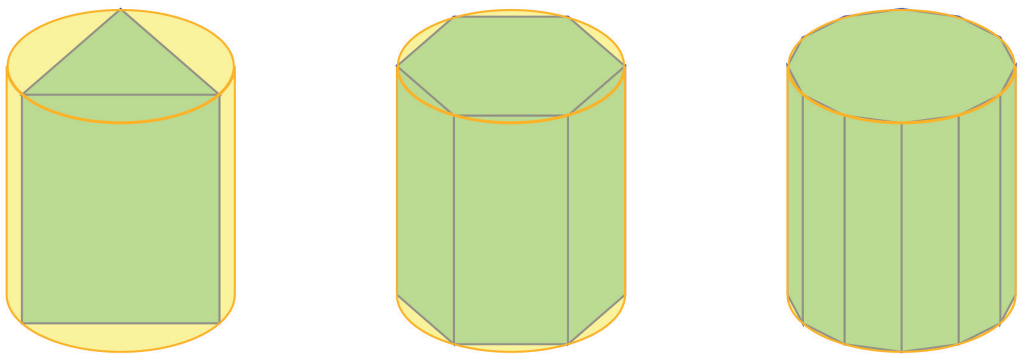
Such objects are called *cylinders*. How do we compute the volume of a cylinder? Is it the product of the base area and the height?

We can draw cylinders in GeoGebra, just as we drew prisms. Draw a circle in the Graphics window and click on it shown in the 3D Graphics window after selecting the Extrude to Prism tool. Enter the height to draw the cylinder.

Recall how we computed the area of a circle by considering regular polygons of increasing number of sides within it:?

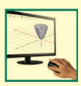


In the same way we can imagine prisms with regular polygons as bases inside a cylinder:



The volumes of these polygonal prisms get closer and closer to the volume of the cylinder, right?

We can now use some algebra to show that the volume of the cylinder is also the product of the base area and height



In GeoGebra, draw a circle with centre at A and radius 3, and mark a point B on it. Make an integer slider n with Max:200. Select the Angle with Given Size tool and click on B and then on A. Give the Angle as (360/n)°. We get a new point B' on the circle. Select the Regular Polygon tool; and click on B and B'. Give the Vertices as n. We get a regular polygon of n sides within the circle. Now in the 3D Graphics window, draw a prism with this polygon as base. See what happens when we increase the value of n.

- Let's denote the base areas of the polygonal prisms by $p_1, p_2, p_3, \ldots$ and so on
- Let's denote the base area of the cylinder by c. Then the numbers $p_1, p_2, p_3, \ldots$ get closer and closer to the number c
- Let's denote the height of the cylinder by h
- The numbers $p_1h, p_2h, p_3h, \ldots$ get closer and closer to the number ch

* The numbers $p_1h, p_2h, p_3h, \ldots$ are the volumes of the polygonal prisms.

* The volumes of the polygonal prisms get closer and closer to the volume of the cylinder.

- Let's denote the volume of the cylinder by v
- The numbers p_1h, p_2h, p_3h, ... get closer and closer to the number v

From the two statements within red boxes above, we get

$$v = ch$$

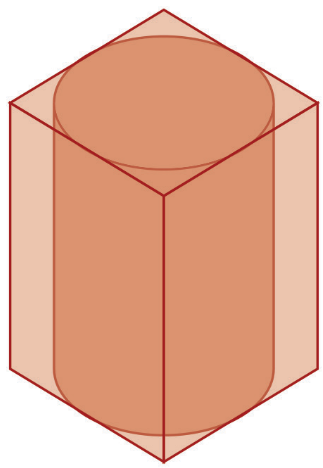
Thus we get this:

The volume of a cylinder is the product of the base area and height.

We know that the area of a circle is the product of the square of the radius by π. So, if the radius of the base of a cylinder is 3 centimetres and its height is 5 centimetres, then its volume is $\pi \times 3^2 \times 5 = 45\pi$ cubic centimetres.

Let's look at another problem:

A block of wood in the shape of a square prism has base edges of length 10 centimetres and height 20 centimetres. What is the volume of the largest cylinder that can be cut out from it?



The base of the largest cylinder possible is the largest circle that can be enclosed with the base of the square pyramid; and the height of this cylinder is the height of the square prism itself:

This means the base diameter of the cylinder should be equal to the side of the base of the square prism.

Thus the base radius of the cylinder is 5 centimetres, so that the base area is 25π square centimetres. Since the height of the cylinder is 20 centimetres, its volume is $25\pi \times 20 = 500\pi$ cubic centimetres

(1) The base diameter of a cylindrical tank is 1 metre and its height is 2 metres. How many litres of water can it contain?

(2) The base radius of an iron cylinder is 15 centimetres and its height is 32 centimetres, It is melted and recast into a cylinder of base radius 20 centimetres. What is the height of this cylinder?

(3) The base radii of two cylinders of the same height are in the ratio 3 : 4. What is the ratio of their volumes?

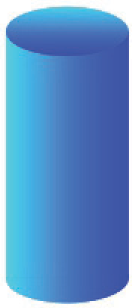
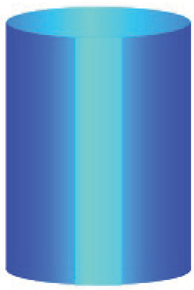
(4) The base radii of two cylinders are in the ratio 2 : 3 and their heights are in the ratio 5: 4

(i) What is the ratio of their volumes?

(ii) The volume of the smaller cylinder is 720 cubic centimetres. What is the volume of the larger one?

Curved surface

A rectangular sheet of paper or metal can be rolled into a cylinder; on the other hand, a hollow cylinder, open at both ends, can be cut and spread out into a rectangle:


The area of this rectangle is called the *curved surface area* of the cylinder.

The length of one side of this rectangle is the height of the cylinder. The other side is the base circle straightened, so that its length is the circumference of this circle. The curved surface area is the product of these lengths:

The curved surface area of a cylinder is the product of its base circumference and its height

The base circumference is π times the base diameter, isn't it? So, the curved surface area of a cylinder of base radius 3 centimetres and height 5 centimetres is $\pi \times 6 \times 5 = 30\pi$ square centimetres.

If this is a closed cylinder, to get the total surface area, we must add to this, the areas of the two circles at the ends; that is, $30\pi + (2 \times \pi \times 3^2) = 48\pi$ square centimetres.

(1) In a school building there are 12 cylindrical pillars, each of base diameter $\frac{1}{2}$ metre and height 4 metres.

(i) What is the curved surface area of a pillar?

(ii) What is the total cost of painting all the pillars at 80 rupees per square metre?

(2) The drum of a road roller has base diameter 80 centimetres and length 1.2 metres:

What is the area of the road levelled when it rolls once?

(3) The curved surface area of a cylinder is equal to its base area. What is the relation between its base radius and height?

(4) A rectangular sheet of metal with sides 48 centimetres and 25 centimetres is rolled into a cylinder of height 25 centimetres and its ends are closed with exactly fitting circles. What is the total surface area of this cylinder?

Rectangular sheets of thick paper, of sides 24 centimetres and 18 centimetres are bent along the longer side to make hollow prisms with bases an equilateral triangle, a square, a regular hexagon, a regular octagon. One of them is rolled into a cylinder. All the prisms have the same lateral surface area and it is also equal to the curved surface area of the cylinder. What about their volumes? Do you note anything special?

What if these shapes are made by bending or rolling along the shorter side of the rectangle? Find out the similarities and differences between these and the first ones, in lateral surface area, curved surface area and volume.

13 Polynomial Pictures

First degree polynomials

We have seen in class 6 how the relation between certain measurements can be written as algebraic equations; and then saw these as operations on pure numbers in the lesson **Polynomials**. Seen this way, every polynomial transforms numbers according to some definite rule.

For example, let's take the first degree polynomial

$$p(x) = 2x + 1$$

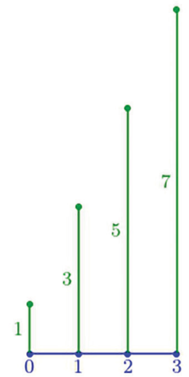
This can be seen as the operation of multiplying every number by 2 and adding 1. So, if we take x as 0, 1, 2, 3 and so on, we get the numbers 1, 3, 5, 7 and so on as $p(x)$. (Recall the general form of odd numbers, seen in class 7)

We can show how $p(x)$ changes with x using a table:

x	0	1	2	3
$p(x)$	1	3	5	7

We can show this as a picture, as done in classes 5 and 6.

First draw a horizontal line and mark equally spaced points on it, labelling them 0, 1, 2, 3 and so on. Now from these points draw perpendiculars of heights $p(0)$, $p(1)$, $p(2)$, $p(3)$ and so on, with the same unit of length used on the horizontal line:

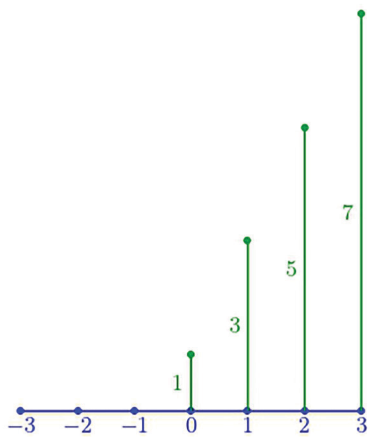


We can take negative numbers also as x, and extend the table to the left:

x	−3	−2	−1	0	1	2	3
$p(x)$	−5	−3	−1	1	3	5	7

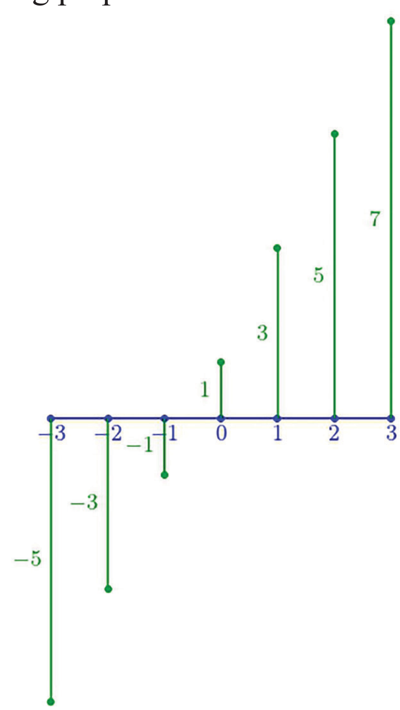
How do we show them in the picture ?

As done in the lesson, **Real Numbers**, we can extend the horizontal line to the left and mark the negative numbers taken as x:



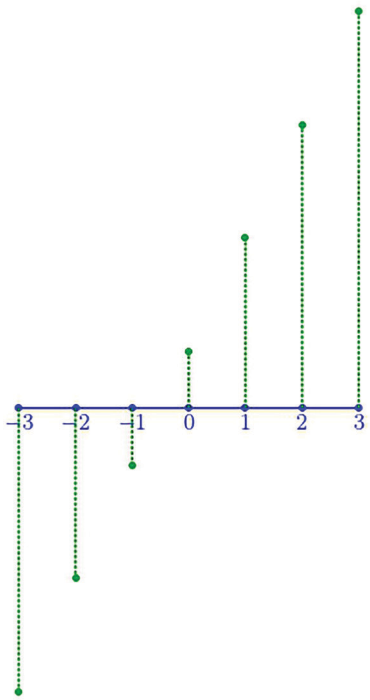
How do we mark negative numbers got as $p(x)$?

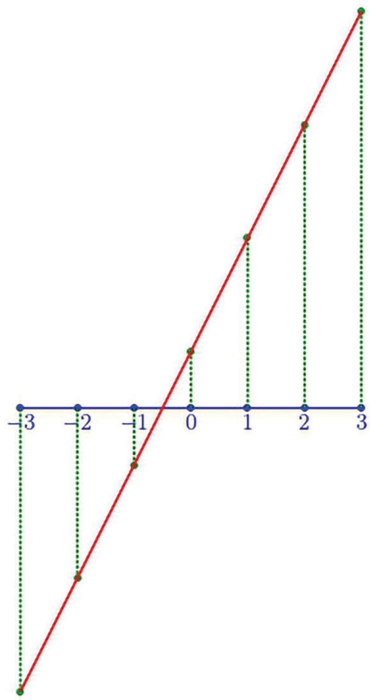
We showed positive values of $p(x)$ by drawing perpendiculars upward; so, let's show negative values of $p(x)$ by drawing perpendiculars downwards:



Now look at the ends of these perpendiculars

Are they all on the same line? Join them using a ruler:



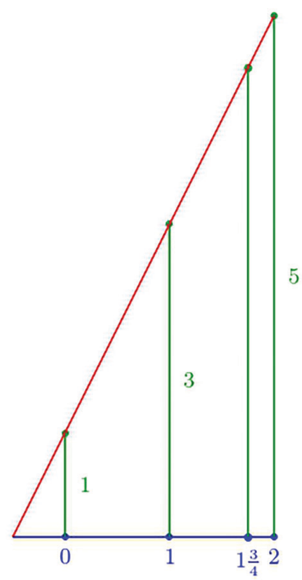


If we take other numbers as x, will the ends of the perpendiculars of lengths $p(x)$ be still on this line?

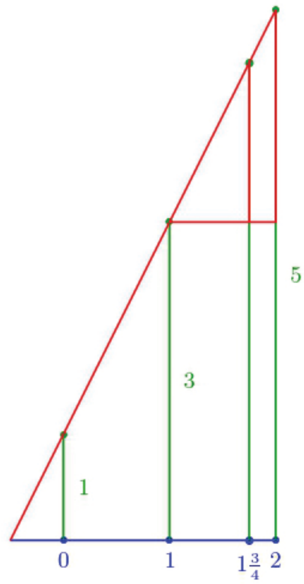
What if we take $x = 1\frac{3}{4}$?

$$p\left(1\frac{3}{4}\right) = \left(2 \times 1\frac{3}{4}\right) + 1 = 4\frac{1}{2}$$

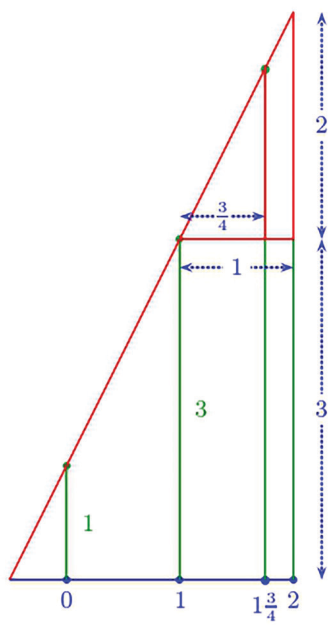
We can mark $1\frac{3}{4}$ on the horizontal line and draw a perpendicular to meet the slanted line. We want to check whether its height is $4\frac{1}{2}$. To see this, we zoom in onto the part of the picture we are interested in:



If we draw another perpendicular as in the picture below, we get two right triangles (shown in red) at the top:

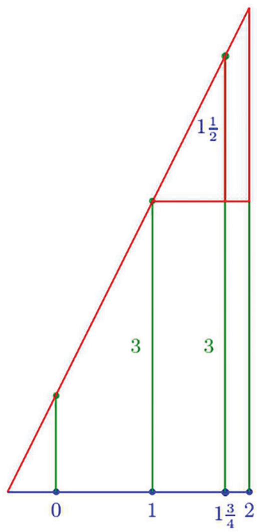


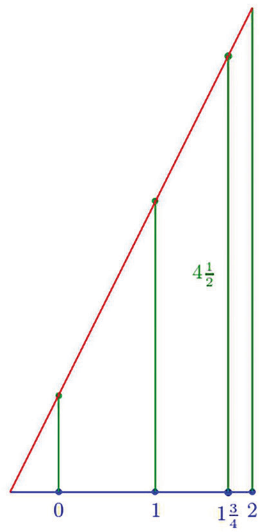
These two triangles have the same angles (why?) and so their sides are scaled by the same factor, as seen in the lesson, **Similar Triangles**. The lengths of some of the sides of these triangles are easily seen:



The bottom side of the smaller triangle is $\frac{3}{4}$ of the bottom side of the larger triangle. So, the vertical side of the smaller triangle must also be $\frac{3}{4}$ of the vertical side of the larger triangle.

So, the length of the vertical side of the smaller triangle is $2 \times \frac{3}{4} = 1\frac{1}{2}$



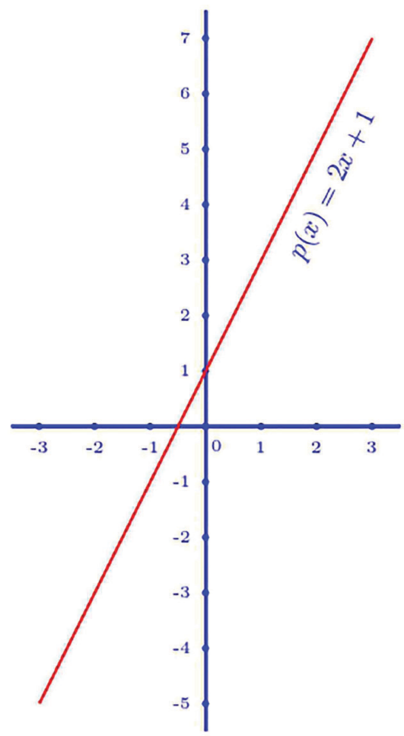


This gives the length of the perpendicular from the point $1\frac{3}{4}$ on the horizontal line to the red as $3 + 1\frac{1}{2} = 4\frac{1}{2}$.

Turning this statement around, we can say that if from the point $1\frac{3}{4}$ on the horizontal line, we draw a perpendicular of height $p\left(1\frac{3}{4}\right) = 4\frac{1}{2}$ then its end is on the red line.

In the same way, we can see that whatever number (rational or irrational) we take as x, the end of the perpendicular of length $p(x)$, drawn from the point corresponding to x on the horizontal line, would be on the red line.

This line is called the *graph* of the polynomial $p(x) = 2x + 1$. Usually when we draw graphs like this, only the ends of the perpendicular are marked (instead of drawing the whole perpendicular) and then these points are joined. To see the heights, a perpendicular to the horizontal line is drawn through the point denoting 0, and distances marked on it with respect to the same unit used in the horizontal line.



We can use the Input Bar in GeoGebra to draw graphs. To draw the graph of $p(x) = 2x + 1$, type $p(x) = 2x + 1$ as Input and hit the Enter key. [We can simply type $2x + 1$ to get the graph, but the name of the polynomial may not be $p(x)$]

In the same way, we can see that the graph of any first degree polynomial is a straight line.

We need only two points to draw a line. So to draw the graph of a first degree polynomial, we need only take two numbers as x and compute the numbers got from the polynomial

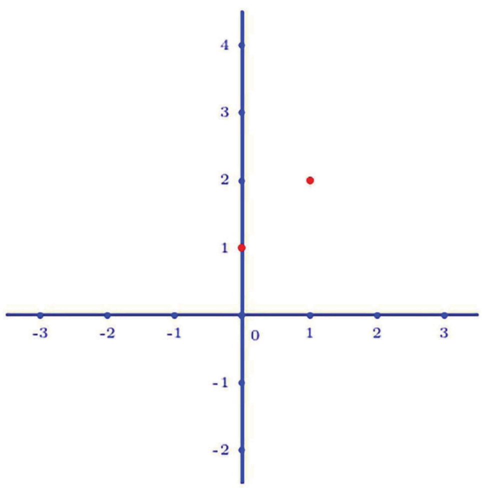
For example, consider the polynomial

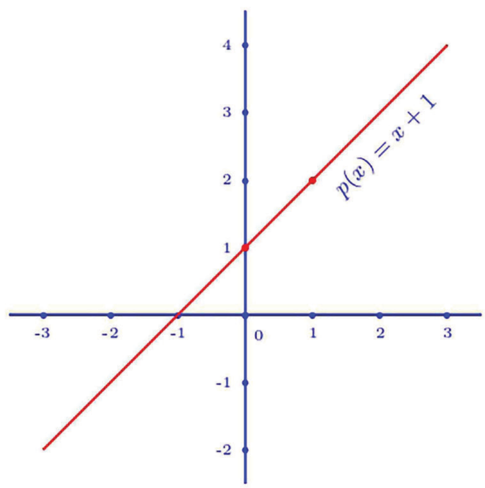
$$p(x) = x + 1$$

Let's take 0 and 1 as x:

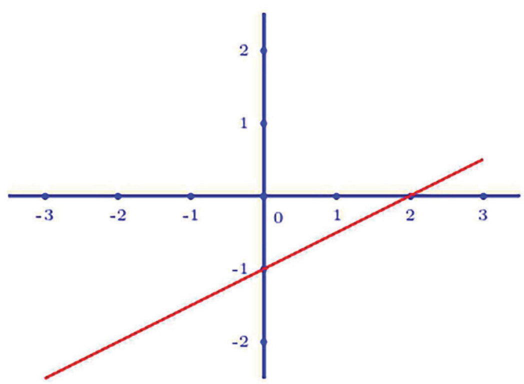
x	0	1
$p(x)$	1	2

Now can't we draw the graph?





On the other hand, it is not difficult to get the first degree polynomial from its graph. For example see this graph:



Let's denote by $p(x)$, the polynomial with this graph. Then we can see from the picture that $p(0) = -1$ and $p(2) = 0$ from the graph.

> Create two sliders a and b in GeoGebra with Increment 0.1. Type p(x) = ax + b in the input bar to draw the graph of this polynomial. Change only b. What happens to the graph? What if we change a only? Can we find out what $p(x)$ is, if $p(0) = -1$ and $p(2) = 5$ using this? Try! What if $p(1) = 2$ and $p(2) = 6$?

Now since $p(x)$ is a first degree polynomial, it must be of the form

$$p(x) = ax + b$$

We want to find out a and b.

For that we use the facts that $p(0) = -1$ and $p(2) = 0$, got from the graph.

Since $p(x) = ax + b$ we have

$$p(0) = (a \times 0) + b = b$$

Since $p(0) = -1$ we get

$$b = -1$$

Using this, we can find $p(2)$ as

$$p(2) = (a \times 2) + b = 2a + b = 2a - 1$$

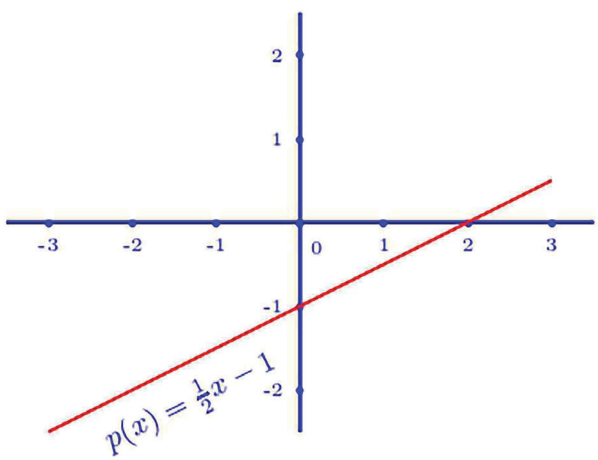
and we have got $p(2) = 0$ from the graph. So,

$$2a - 1 = 0$$

which means

$$a = \frac{1}{2}$$

So, the polynomial of the graph is $p(x) = \frac{1}{2}x - 1$. In other words, the graph of $p(x) = \frac{1}{2}x - 1$ is this line:

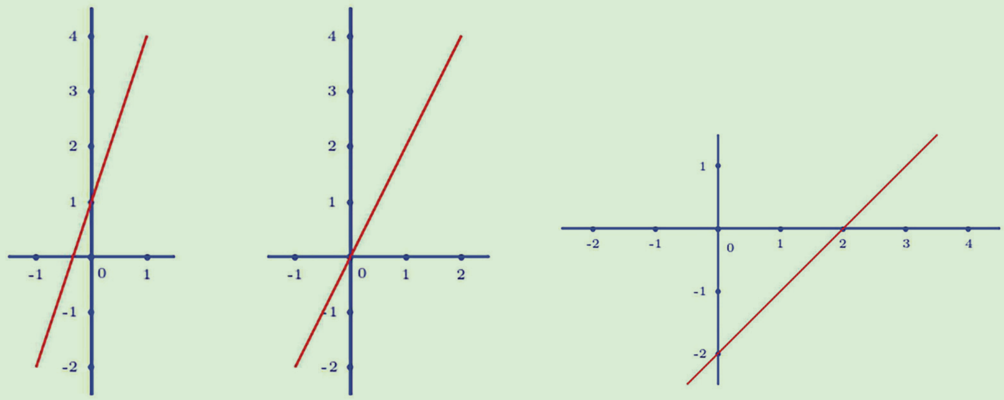


(1) Draw the graphs of these polynomials:

(i) $p(x) = 2x - 1$ (ii) $p(x) = x - 1$ (iii) $p(x) = 1 - x$

(iv) $p(x) = x$ (v) $p(x) = -x$

(2) Find the polynomials which has these lines as their graphs:



(i) (ii) (iii)

Second degree polynomials

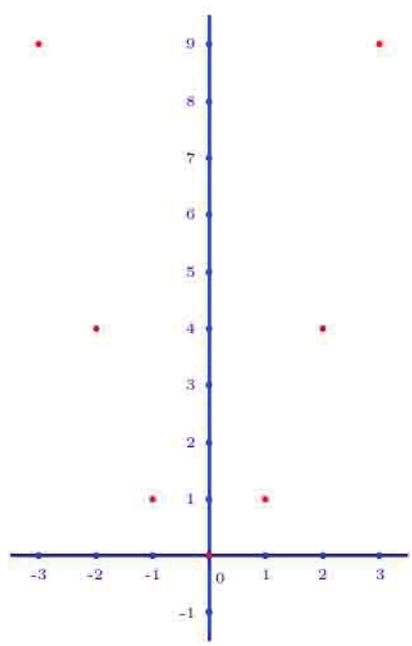
Let's look at graphs of second degree polynomials next. What is the simplest second degree polynomial?

$$p(x) = x^2$$

To draw its graph first let's make a small table:

x	−3	−2	−1	0	1	2	3
$p(x)$	9	4	1	0	1	4	9

Now as before, we draw two lines perpendicular to each other and mark equally spaced points on it, labelling them with numbers. Next, above each number taken as x on the horizontal line, mark the point at a height $p(x)$

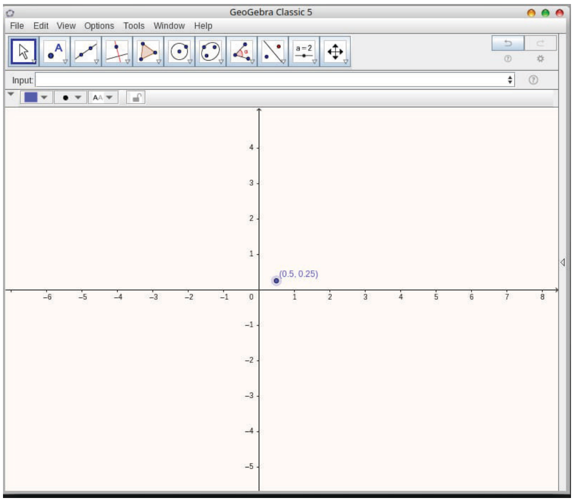


Using these points only, we cannot draw the graph; how do we get some points on the graph in between these?

If we take $x = 0.5$, then $p(0.5) = 0.25$ and this distance we cannot mark with a ruler.

We can use GeoGebra to draw the graph.

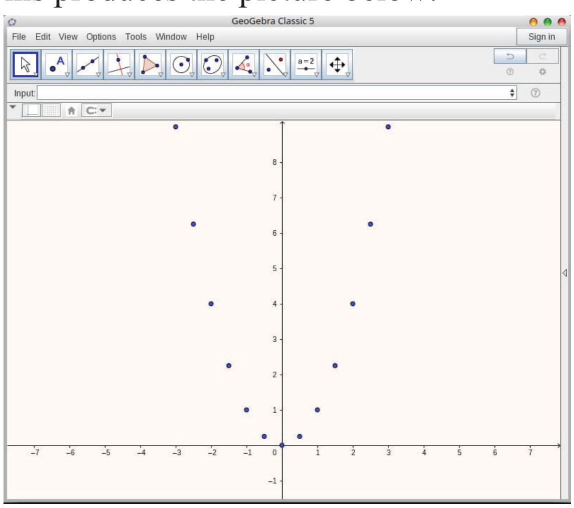
On opening GeoGebra, we can see horizontal and vertical lines marked with numbers. If we type (0.5,0.25) in the input bar, we get the point at a height 0.25 above 0.5 on the horizontal line.



Instead of separately computing the distance and height of each point we want and marking them one by one, we can do it in one go in GeoGebra. Type the following command in the Input Bar:

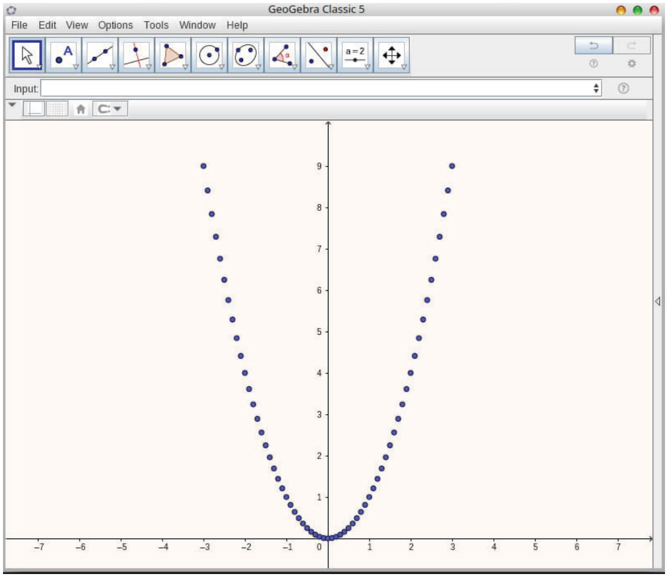
```
Sequence [(t, t^2), t, -3, 3, 0.5]
```

This asks GeoGebra to take as t the numbers starting from -3 and repeatedly increased by 0.5 till 3 is reached, and then for each such t, mark the point at a height t^2 above t on the horizontal line. This produces the picture below:



To get more points in between all we need to do is change the difference between the numbers from 0.5 to 0.1:

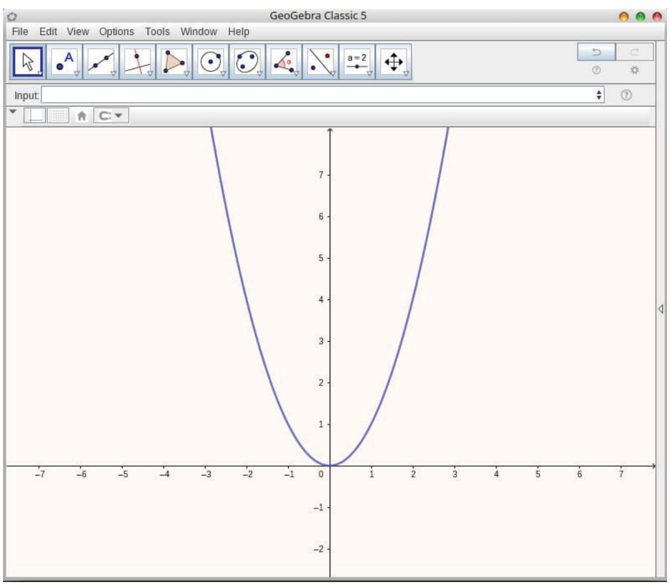
```
Sequence[(t,t^2),t,-3,3,0.1]
```



> In GeoGebra, create a slider a with Min:0, Max:1 and Increment:0.01 Type Sequence [(t, t^2), t, −3, 3, a] in the inputbar.
> Some points on the graph of $p(x) = x^2$ will be shown. As we decrease the value of a, we get more and more points on the graph.

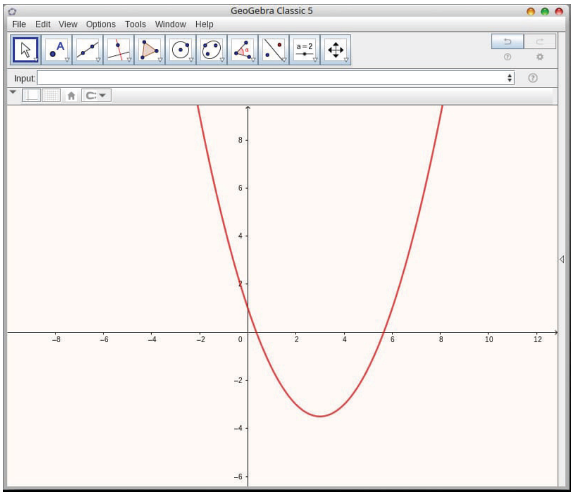
Now don't we get an approximate shape of the graph?

To draw this curve without in gaps, we type **x^2** in the Input Bar.

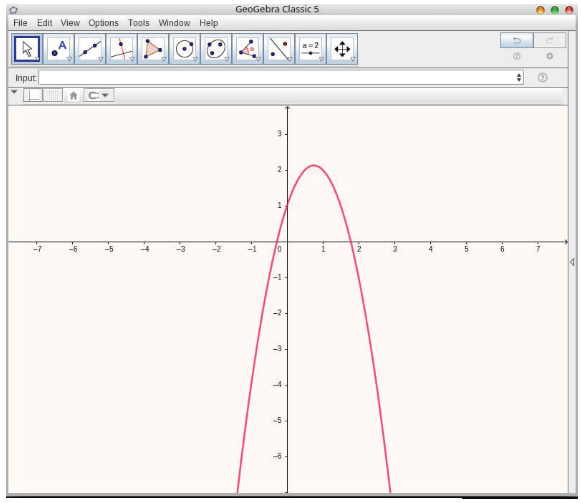


> In GeoGebra, create a slider a with Increment:0.1. If we type (a, a ^ 2), we get a point on the graph of $p(x) = x^2$. As we change the value of a, using the slider, the point moves. To see the path of its travel, right click on the point and check Trace on in the menu. If we decrease the increment to say, 0.01, we get points closer together. (We can change the Increment by right clicking on the slider and selecting Object Properties → Slider → Increment). We can get the path of travel by selecting the Locus tool and clicking on the slider and the point

Type other second degree polynomials in the Input Bar and see the graphs. For example, the graphs got on typing **(1/2)x^2 − 3x + 1** and **−2x^2 + 3x + 1** are shown on the next page.



$\frac{1}{2}x^2 - 3x + 1$



$-2x^2 + 3x + 1$

Looking at the graphs of various second degree polynomials, we note that the only differences are change in position, change in spread or a vertical turnabout.

When we look at these curves from left to right, we see two different kinds of behaviour:

- Moving down to a minimum and then moving up.
- Moving up to a maximum and then moving down

How do we describe these using algebra?

In the second degree polynomial $p(x)$, if the numbers taken as x are gradually increased from negative numbers to positive numbers, the numbers $p(x)$ behave in either of these two ways:

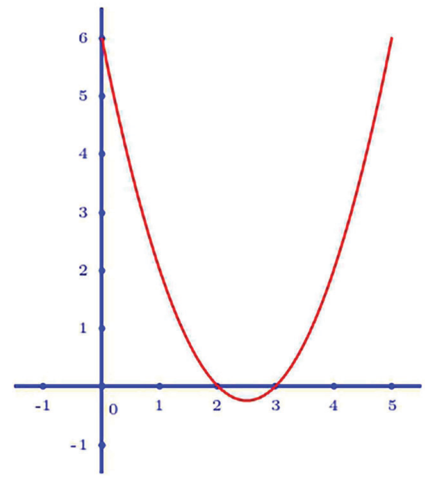
- The numbers $p(x)$ decrease to a minimum and then increase
- The numbers $p(x)$ increase to a maximum and then decrease

In GeoGebra, create a slider a and draw the graph of the polynomial `p(x) = x^2 + a`. What happens to the curve as the number a is changed?

Next draw the graph of `q(x) = ax^2`. In this, what is the change, when the number a is changed?

Now let's see how we can find a second degree polynomial from its graph.

For example, the picture shows the graph of a second degree polynomial. Denoting the polynomial by $p(x)$, we see from the picture that



$$p(0) = 6$$

$$p(2) = 0$$

$$p(3) = 0$$

Since $p(x)$ is a second degree polynomial, we can write it as

$$p(x) = ax^2 + bx + c$$

as seen in the lesson, **Polynomials**

So

$$p(0) = (a \times 0^2) + (b \times 0) + c = c$$

Now since $p(0) = 6$ we get

$$c = 6$$

and so

$$p(x) = ax^2 + bx + 6$$

Next from $p(2) = 0$ we get

$$4a + 2b + 6 = 0$$

and from $p(3) = 0$ we get

$$9a + 3b + 6 = 0$$

We can rewrite the first equation as

$$2a + b = -3 \qquad (1)$$

and the second equaiton as

$$3a + b = -2 \qquad (2)$$

Subtracting Equation (1) from Equation (2) gives

$$a = -2 -(-3) = -2 + 3 = 1$$

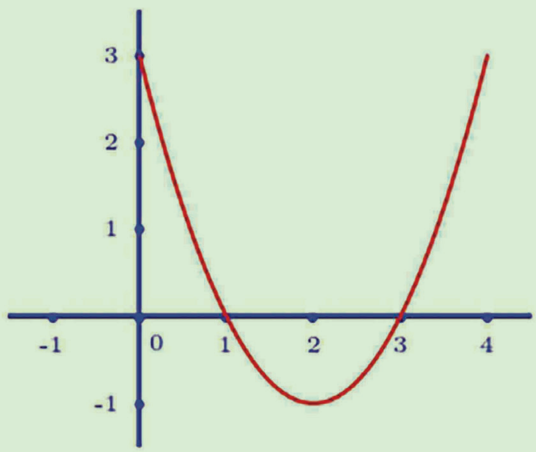
Using this in Equation (1) gives

b = □3 □ 2 = −5

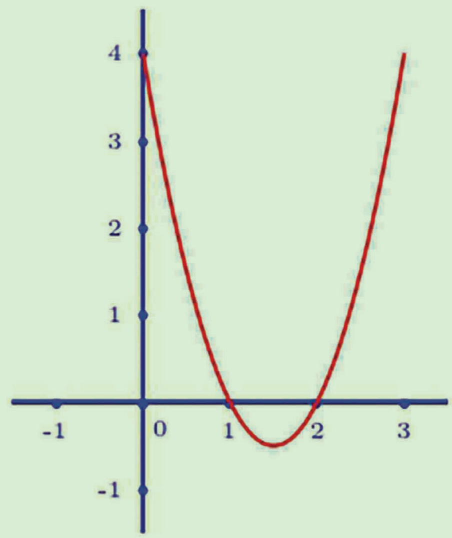
So, the polynomial of this graph is

$p(x) = x^2$ □ $5x + 6$

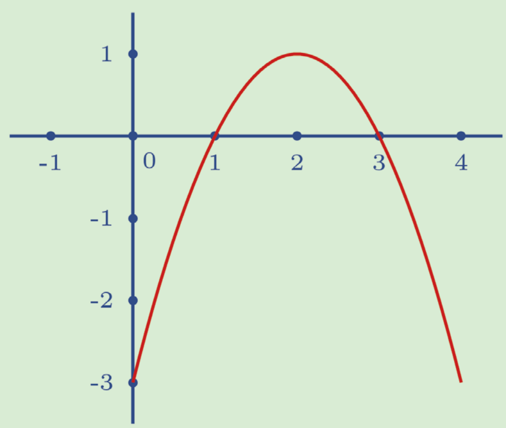
The graphs of some second degree polynomials are given below:



(i)



(ii)

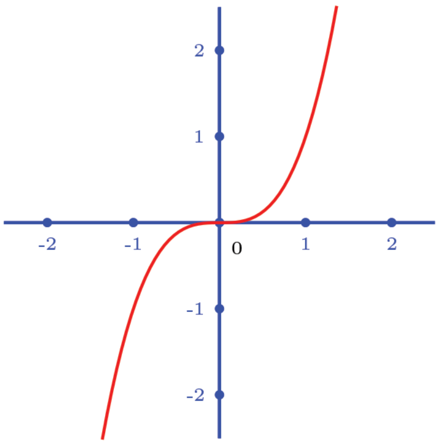


(iii)

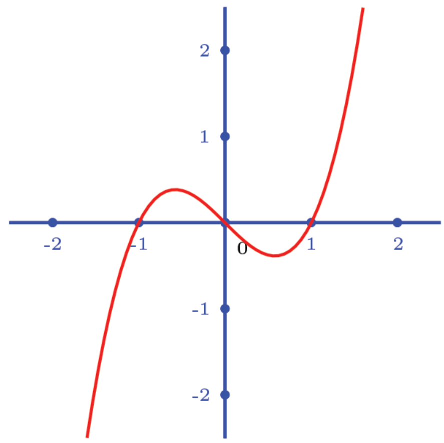
Find the polynomials

Graphs of third degree polynomials do not have such a common form. For example, draw the graphs of these polynomials in GeoGebra and see:

- $p(x) = x^3$
- $p(x) = x^3 - x$



$p(x) = x^3$



$p(x) = x^3 - x$

Draw the graphs of these polynomials::

(i) $p(x) = x$ (ii) $p(x) = 2x$ (iii) $p(x) = \frac{1}{2}x$ (iv) $p(x) = -x$ (v) $p(x) = -2x$

Do you see any common feature of these graphs?

Now draw the graphs of these polynomials:

(i) $p(x) = x + 1$ (ii) $p(x) = x + 2$ (iii) $p(x) = x + \frac{1}{2}$ (iv) $p(x) = x - 1$
(v) $p(x) = x - 2$

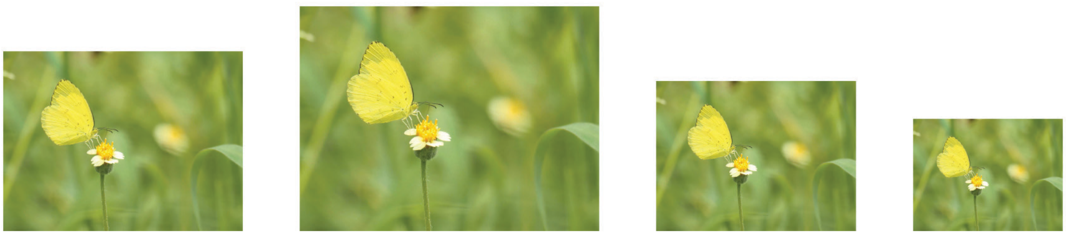
Do you see any common feature of these graphs?

14

PROPORTION

Proportional changes

See these pictures:



The same photographs in different sizes.

We have noted in the lesson, **Ratios** in class 7 that, in enlarging or reducing a photo like this, the ratio of the sides of the rectangle should not change.

In all the pictures above the width to height ratio is 4 : 3. That is, the height is $\frac{3}{4}$ of the width; or in reverse, the width is $\frac{4}{3} = 1\frac{1}{3}$ of the height.

Now look at these pictures:

A photo in the 16 : 9 format in different sizes. This means that in all these figures, the height is $\frac{9}{16}$ of the width, or in other words, the width is $\frac{16}{9} = 1\frac{7}{9}$ of the height.

Thus in each set of pictures, though the width and height change, one is a fixed multiple or fraction of the other.

Let's state this using algebra.

In the first set of photos, if we denote the width by w and the height by h, then in all these,

$$h = \frac{3}{4}w$$

Using the same notation for the second set, we have

$$h = \frac{9}{16}w$$

There are several instances where two varying quantities are related like this. For example, if we draw circles of different radii, in all these the circumference would be 2π times the radius, right? Denoting the radius of these by r and the circumference by c, we have

$$c = 2\pi r$$

In the examples so far, we have been considering only relations between lengths. This kind of relation occurs between different kinds of quanti ties also.

For example, if an object travels along a straight line at the same speed of 10 metres per second, then the distance travelled at any instant is 10 times the time of travel. If we denote the time of travel as t seconds and the distance travelled as s metres, then

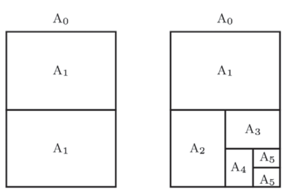
$$s = 10t$$

There's a general name for varying quantities related in this manner:

If two related quantities change in such a way that the number representing one quantity is a fixed multiple or fraction of the other, then the change is said to be *proportional.*

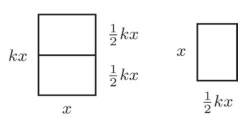
Paper proportions

For printing and copying, we now mostly use A_4 size paper. This is just one of a series of decreasing paper sizes, starting with A_0 and continued to A_1, A_2 and so on. The size of each such sheet in the series is half that of the preceding one:



Also, in all such sheets, the width to height ratio is the same

Let's denote the widths of such sheets by x and the height by kx. The width and height of a sheet in the next type would be then $\frac{1}{2}kx$ and x:



Since the width to height ratio is the same for both, we have

$$\frac{1}{2}k = \frac{1}{k}$$

From this, we get $k^2 = 2$ which means

$$k = \sqrt{2}$$

Thus in the A-series sheets of paper, height is proportional to the width and the proportionality constant is $\sqrt{2}$.

(The word *proportion* is generally used to indicate equality of ratios)

In proportional changes of two quantities, the fixed number which gives one quantity as a multiple or fraction of the other is called the *proportionality constant.*

Let's have another look at our earlier examples:

- If we use a program like GIMP to change the width of any of our first set of pictures, then on entering the desired width, the program itself would calculate the height as $\frac{3}{4}$ of the width and make the changes. Here, height changes proportionally with respect to width and the proportionality constant is $\frac{3}{4}$.

  On the other hand, if we instruct the height to be changed, then the width would be calculated as $\frac{4}{3}$ of the height entered. In this case, width changes proportionally with respect to height and the proportionality constant is $\frac{4}{3}$.
- If the pictures are in the 16 : 9 format, as in the second set of pictures above, then a graphics program calculates height proportionally with respect to the width, with proportionality constant $\frac{9}{16}$, and calculates width proportionally with respect to height, with proportionality constant $\frac{16}{9}$.
- If circles of different radii are drawn, the circumference changes proportionally with respect to radius and the proportionality constant is 2π.

  On the other hand if wires of different lengths are bent into circles, the length of the wire is the circumference of the circle; and the radius changes proportionally with respect to this length and the proportionality constant is $\frac{1}{2\pi}$.
- For an object moving along a straight line at the constant speed of 10 metres per second, the distance travelled (in metres) changes proportionally with respect to the time of travel (in seconds) and the proportionality constant is 10.

  On the other hand, the time of travel (in seconds) changes proportionally with respect to the distance travelled (in metres) and the proportionality constant is $\frac{1}{10}$.

We can describe proportional changes using algebra:

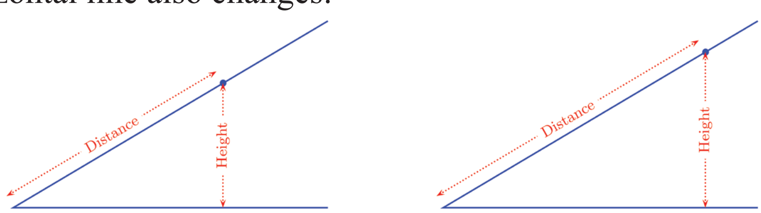
If the relation between two quantities measured as the numbers x and y is

$$y = kx \text{ where } k \text{ is a constant,}$$

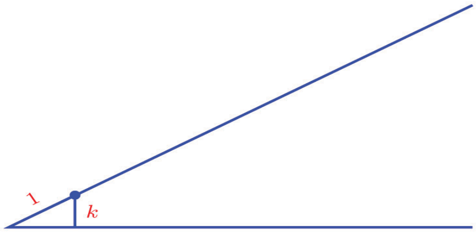
then the quantity indicated by y changes proportionally with respect to the quantity indicated by x; and the proportionality constant is k

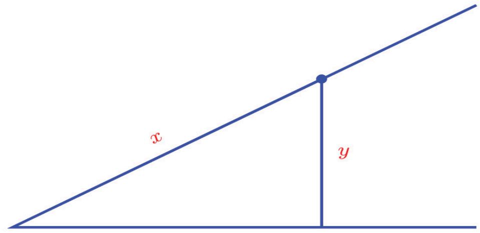
Let's look at another example:

For all points on the slanted line, as the distance from the corner changes, the height from the horizontal line also changes:



To check whether this change is proportional, let's denote by k, the height of a point at a distance of 1 from the corner:





As in the second picture, we denote the distance from the corner by x, and the height above the horizontal line by y.

Since the right triangles in the two pictures have the same angles, their sides are scaled by the same factor, as seen in the lesson, **Similar Triangles**. This means

$$\frac{x}{1} = \frac{y}{k}$$

which gives

$$y = kx$$

So, the height above the horizontal line changes proportionally with respect to the distance from the corner.

(1) In each of the instances below, show that the second quantity changes proportionally with respect to the first.

Also find the proportionally constant in each:

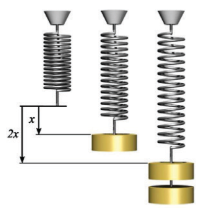
(i) The length of sides of squares and their perimeters.

(ii) The lengths of wires bent into squares and the length of the sides of the squares.

(iii) The number of rotations of a circle rolling along a line, and the distance travelled along the line.

Weight and length

When a weight is suspended from a spring, its extension is proportional to the weight:

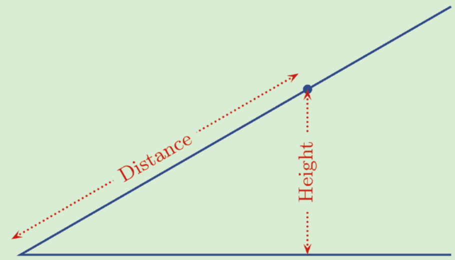


This was discovered by Robert Hooke, an English physicist of the seventeenth century.

This principle can be used to calibrate spring scales:

If a weight of 1 kilogram extends the spring by 5 centimetres, then the extension for 100 grams would be $\frac{1}{2}$ centimetre, right? So if the scale is marked at $\frac{1}{2}$ centimeter gaps, it can measure weights which are integer multiples of 100 grams. If smaller intervals are marked, then weights less than 100 grams and their multiples can also be weighed.

(2) We have seen that in the picture, the height of a point on the slanted line from the horizontal line changes proportionally with respect to its distance from the corner. Calculate the proportionality constants when the angle is 30°, 45° and 60°.

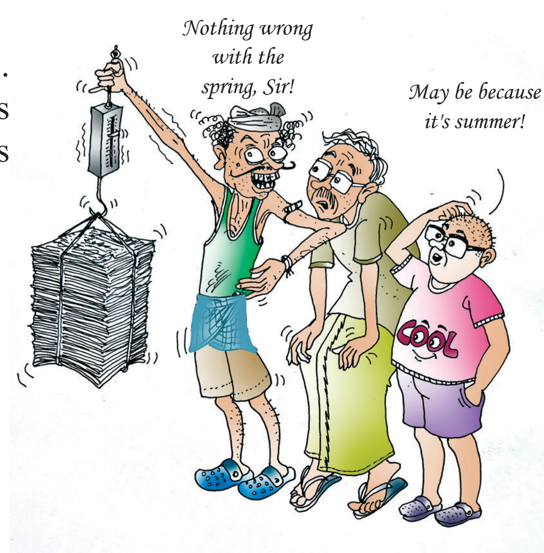


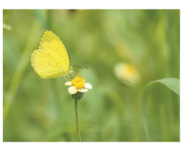
(3) Prove that in equilateral triangles, the perimeter changes proportionally with respect to the length of the sides. What is the proportionality constant? What can we say about other regular polygons?

(4) Prove that the lengths of arcs of a fixed circle change proportionally with respect to their central angles. What is the proportionality constant? What about the relation between the area of a sector and its central angle?

Scale and proportion

Let's look at a property of proportional change. If the width of one of the photos we first saw is doubled using a graphics program, what happens to the height?





In both photos, height is $\frac{3}{4}$ of the width. So,

$$\text{New height} = \frac{3}{4} \times \text{New width}$$

And what is the relation between the old and new widths?

$$\text{New width} = 2 \times \text{Old width}$$

So,

$$\text{New height} = \frac{3}{4} \times 2 \times \text{Old width}$$

$$= 2 \times \frac{3}{4} \times \text{Old width}$$

In this,

$$\frac{3}{4} \times \text{Old width} = \text{Old height}$$

isn't it? So,

$$\text{New height} = 2 \times \text{Old height}$$

Thus if the width is doubled, then height also is doubled.

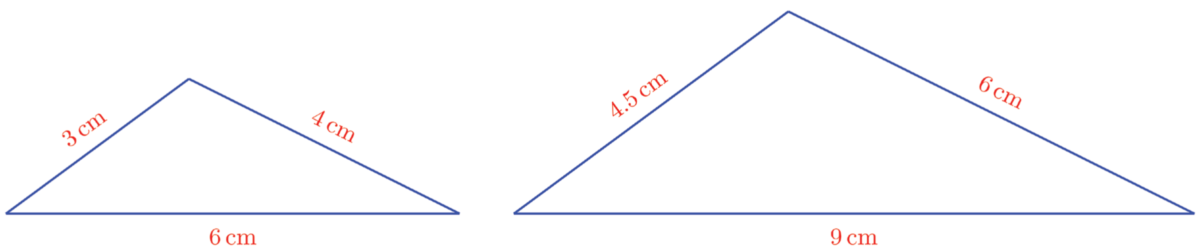
In the same way, we can see that if the width is tripled, the height is also tripled and if the width is halved, then the height is also halved. In general, the change in height is the same multiple or fraction of what the width is changed into. In other words, height is scaled by the same factors as width. (But in all these, the proportionality constant is the change of height with respect to width is $\frac{3}{4}$ itself.)

It is not difficult to see this is true in other instances of proportional changes.

In proportional changes of quantities, one is scaled by the same factor as the other

This can be used to check whether some changes are proportional. For example, an object dropped from a height travels 4.9 metres in one second and 19.6 metres in two seconds. In other words, when time is doubled, distance is not doubled, but quadrupled. So, in this travel, distance does not change proportionally with respect to time.

We have seen in the lesson **Similar triangles** that in triangles with the same angles, sides are scaled by the same factor; and on the other hand, if the sides of a triangle are scaled by the same factor, then the angles are not changed. For example, see these triangles:

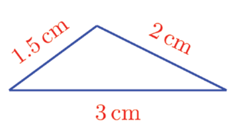


The sides of the larger triangle are $1\frac{1}{2}$ times the sides of the smaller one. In other words, the sides of the small triangle are scaled by a factor of $1\frac{1}{2}$ to get the larger triangle.

We can note another thing here. In both triangles. we have these relation between pairs of their sides:

- The longest side is 2 times the smallest side
- The medium-length side is $1\frac{1}{3}$ times the shortest side
- The longest side is $1\frac{1}{2}$ times the medium-length side

These relations do not change even if we scale the triangle by any other factor. For example, see this picture:



This is drawn by halving the sides of the first triangle by $\frac{1}{2}$, or by taking one-third the sides of the second triangle.

So the scale factor for the change from the first triangle to the third triangle is $\frac{1}{2}$ and the scale factor for the change from the second to the third is $\frac{1}{3}$.

But in this triangle also, the ratio of the sides is the same: the longest side is double the smallest side and the medium-length side is $1\frac{1}{3}$ times the smallest. (Because of these, the longest side must be $1\frac{1}{2}$ times the medium-length side).

Thus, the first triangle can be scaled by different factors to get different triangles; but in all these, the ratios of pairs of sides do not change.

This we state as a general result:

> If we consider all triangles with the same angles, the length of each side changes proportionally with respect to another side

Rain math

When it rains, the amount water falling on each square centimeter is the same. In other words the volume of water falling in a region is proportional to its area.

So when it rains, if we put some vessels, cylinders or prism-shaped close together on the the ground, the volume of water collected in each, divided by the base area must be the same number. But the volume of water in a prism or cylinder, divided by the base area is the height of the water level, right? Thus whatever be the size of the vessel we use, the height of the rain water collected would be the same. This height is taken as a measure of the rainfall. According to this, if the rainfall is 1millimetre, then in a tank with base area 1square metre, water will rise to a height of 1millimetre. The volume of this

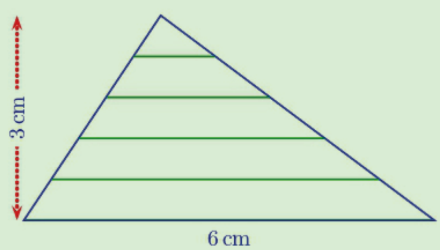
$$100 \times 100 \times 0.1 = 1000$$

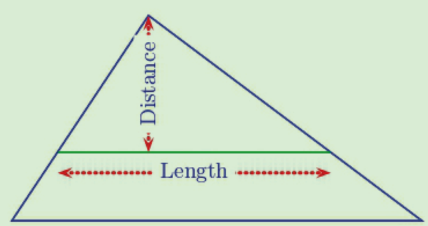
In other words, volume of water is 1 litre. Thus 1millimetre rainfall over a region means 1 litre of rain water in each square metre

(1) (i) What is the sum of the angles of a triangle? And the sum of the angles of a hexagon?

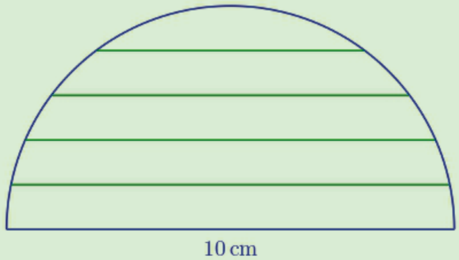
(ii) Does the sum of the angles of polygons change proportionally with respect to the number of sides? Explain the reason.

(2) Inside a triangle of base 6 centimetres and height 3 centimetres, lines are drawn parallel to the base. Prove that the lengths of these lines change proportionally with respect to the distance from the top vertex. Find the proportionality constant.

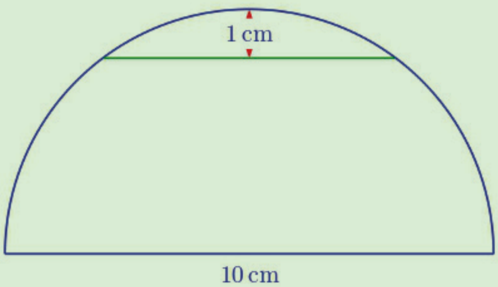


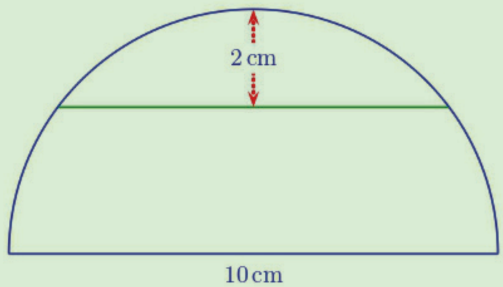


(3) Within a semicircle of diameter 10 centimetres, lines are drawn parallel to the diameter:



(i) In each of the pictures below, calculate the length of the line parallel to the diamenter:





(ii) Does the length of the parallel line change proportionally with respect to the distance from the top of the semicircle. Explain the reason.

Different proportions

In polygons, does the sum of the inner angles change proportionally with respect to the number of sides?

The sum of the angles is 180° for triangles and 720° for hexagons; when the number of sides is doubled, the sum of angles is more than the double. So, the relation is not proportional.

We have seen in the lesson, **Polygons** of class 8, that the sum of the inner angles of any polygon is got by subtracting 2 from the number of sides and then multiplying this number by 180°. That is, if we denote the sum of the angles by s° and the number of sides by n, then

$$s = 180(n - 2)$$

Let's denote the number $n - 2$ here by m. Then the equation becomes

$$s = 180m$$

This shows that quantity s is proportional to the quantity m; that is, the sum of the inner angles is proportional to the number of sides reduced by 2.

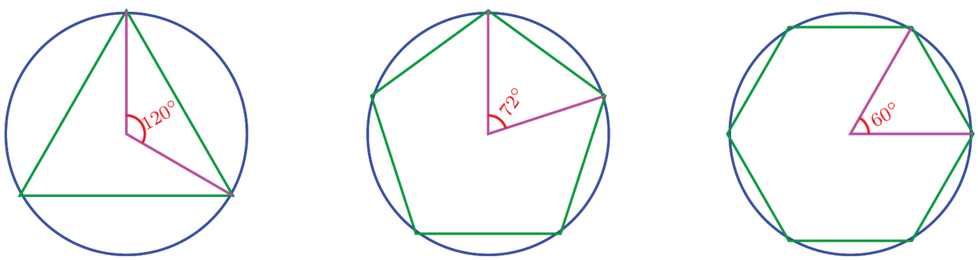
There are many instances like this, where even though a change in two quantities is not proportional, it does become proportional if one of the quantities is changed a bit

For example, since the areas of a circle is the product of the square of the radius and the fixed number π, area is not proportional to the radius; but area is proportional to the square of the radius.

Similarly, though the distance travelled by a falling object is not proportional to time, it is proportional to the square of time.

Let's look at another thing.

For any regular polygon, we can draw a circle through all its vertices. How is the angle made by two adjacent vertices at the centre of the circle related to the number of sides of the polygon?



If we denote the number of sides of the regular polygon by n and denote the central angle as d°, then the relation between them is

$$d = \frac{360}{n}$$

This means, the quantity d changes proportionally with respect to the reciprocal of the quantity n.

Such a change has its own name:

If two varying quantities are related in such a way that the number representing one quantity is a fixed multiple or fraction of the reciprocal of the other, then the change is said to be *inversely proportional.*

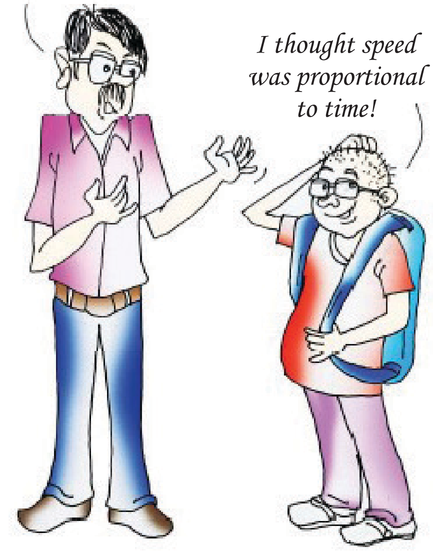
Using algebra, we can state this as follows:

If the relation between two quantities measured as the numbers x and y is

$$y = \frac{k}{x} \text{ where } k \text{ is a constant}$$

then the quantity indicated by y changes inversersely proportional to the quantity indicated by x; and the proporitionality constant is k.

Changes given by the equation $y = kx$ are sometimes called *directly proportional* (instead of simply "proportional", as we have been doing so far) to distinguish it from inversely proportional relationships Let's look at another example of inverse proportionality.



Imagine an object travelling along a straight line from one point to another point 100 metres away. If the speed is 10 metres per second, it takes 10 seconds to reach the other point. If the speed is increased to 25 metres per second, time is reduced to 4 seconds.

If we denote the speed as x metres per second and the time taken to reach the destination as y seconds, the relation between x and y is given by

$$y = \frac{100}{x}$$

So, y changes inversely proportional to x.

(1) (i) Prove that the areas of equilateral triangles change proportionally with respect to the squares of the lengths of sides. What is the proportionality constant?

(ii) Are the areas of squares proportional to the squares of the lengths of sides? If so, what is the proportionality constant?

(2) Consider all rectangles of area 1 square metre.The length of one side of such a rectangle depends on the length of the other side. Write this relation as an algebraic equation. How do we state it in terms of proportion?

(3) Consider all triangles of a fixed area. How do we state in terms of proportion, the relation between the length of the longest side and the length of the perpendicular to it from the opposite vertex? What if we use the shortest side instead of the longest?

(4) In regular polygons, can we say the relation between the number of sides and the measure of an outer angle, in terms of proportion? What is the proportionality constant?

Volume and weight

In objects made of the same material, weight changes proportionally with respect to volume.

As an example, consider objects made of iron. If the weight of any such object is measured in grams and its volume in cubic centimetres, then the number indicating the weight would be 7.87 times the number indicating volume. In other words, 7.87 is the proportionality constant of this change. For objects made of copper, the proportionality constant is 8.96. We say that the density of iron is 7.87 grams per cubic centimetre and the density of copper is 9.86 grams per cubic centimetre.

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Statistics

Average

The table below shows the runs scorerd in the first 10 overs of a one-day cricket match:

Over \ Ball	1	2	3	4	5	6
1	2	1	0	2	0	0
2	1	4	0	1	0	0
3	0	0	4	0	0	1
4	4	0	2	0	1	0
5	0	0	4	1	0	3
6	2	0	2	2	0	0
7	1	0	4	0	2	1
8	2	0	2	0	3	0
9	0	0	6	2	2	0
10	2	0	0	4	0	2

This contains all the details of runs scored in each ball of every over. But we don't get a general view of what happened in the first ten overs.

For that it may be better to condense the table and show only the total runs scored in each over:

Over	1	2	3	4	5	6	7	8	9	10
Runs	5	6	5	7	8	6	8	7	10	8

This table does not have all the details in the first one. But still, we can see certain things at a glance:

- In every over, at least 5 runs were scored
- In eight of these ten overs, more than 5 runs were scored

Next, without drawing up any table, we can just add up all the runs scored in these overs and simply make the statement

70 runs were scored in the first 10 overs

This gives only two numbers. But even with these, we can say something about the batting:

- Not a bad start
- If this trend of scoring nearly 70 runs continues, the total score can be expected to be around 350

We can reduce this to just one number, by dividing 70 by 10 and say this:

The run rate is 7 runs per over

Now what if we don't know any details, but only the run rate?

We cannot conclude that exactly 7 runs were scored in each over; but we can say these:

- If the same number of runs was scored in every over, it would be 7 runs
- The runs scored in every over cannot all be less than 7
    - If that be so, the total would be less than 70
- The runs scored in every over cannot all be more than 7
    - If that be so, the total would be more than 70

Let's look at another situation. There are 40 students in a class. All of them contributed in a fund raising. If we look at the amounts donated by each, there would be just 40 numbers; and from this, we cannot easily see how much money was raised or approximately how much each donated. But suppose just this fact is given instead:

The total contribution of the 40 students is 2000 rupees

It may not be that everyone donated the same amount. So, we cannot say that each of the students gave $2000 \div 40 = 50$ rupees.

Still we can conclude certain facts, as in the case of the cricket match:

- If all students gave the same amount, it would be 50 rupees
- At least one student donated 50 or more (If each had given less than 50 rupees, the total would have been less than 2000 rupees).

- At least one student donated 50 or less (If each had given more than 50 rupees, the total would have been more than 2000 rupees)
- If not all gave the same amount, then some gave more than 50 rupees and some less than 50 rupees

In both these instances, we started with numbers giving a lot of information and reduced them to a single number which gave us a general outline of the information.

To do this, we did the same operation with the numbers in both:

- In the first case, we divided 70, the total number of runs, by 10, the number of overs, to get 7
- In the second, we divided 2000, the total contribution, by 40, the number of students, to get the number 50

Such a number, got by dividing a sum by the number of terms, is called *average* in ordinary language. In the more technical language of mathematics, it is called the *arithmetic mean* or simply the *mean*.

Thus the single numbers calculated in the two examples above can be described as follows:

- The mean of the runs scored in the first 10 overs is 7 (In the first 10 overs, average score is 7 runs an over)
- The mean of the contribution made by 40 students is 50 rupees (Each of the 40 students contributed 50 rupees on average)

We have noted that for unequal numbers, the arithmetic mean is larger than some of these numbers and smaller than some others. We next look at this fact in a little more detail

As an example, let's have a look at the table used to compute the arithmetic mean as 7 in the cricket match:

Over	1	2	3	4	5	6	7	8	9	10
Runs	5	6	5	7	8	6	8	7	10	8

Let's also write how much more or less than the mean was actually scored in each over:

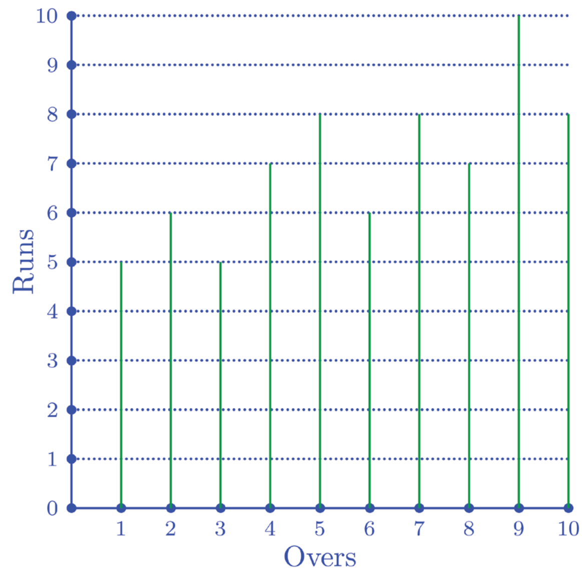
Over	1	2	3	4	5	6	7	8	9	10
Runs	5	6	5	7	8	6	8	7	10	8
Excess over mean					1		1		3	1
Deficit from mean	2	1	2			1				

Let's also add up the excess and deficit separately

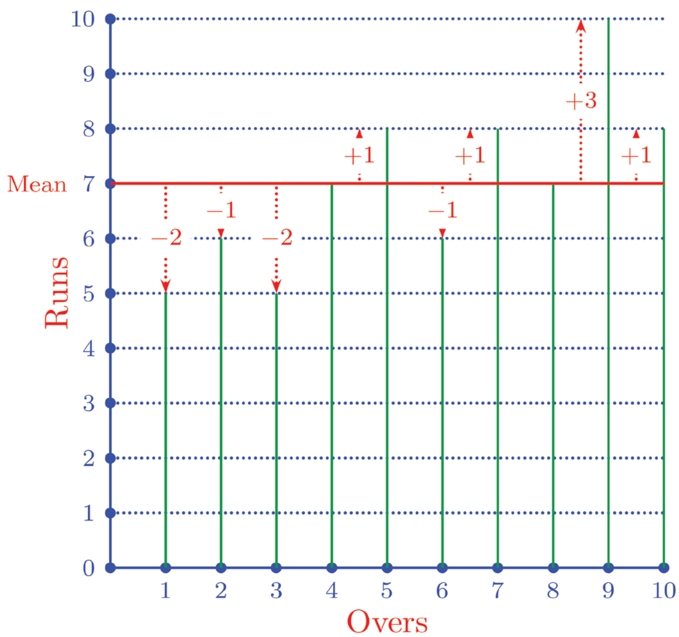
$$\text{Excess} = 1 + 1 + 3 + 1 = 6$$

$$\text{Deficit} = 2 + 1 + 2 + 1 = 6$$

We can also show the runs scored in each over in a picture, as done in classes 5 and 6:



In this, we can draw a horizontal line through the mean 7, and mark the excess and deficit in each over:



Take some numbers and calculate the arithmetic means. Calculate the excess or deficit of each of the numbers from the mean and add them up separately. Are the sums equal? Can you explain the reason for this?

Tables

When numerical data are tabulated, repeated numbers are usually collected together. For example, in a workplace, those getting the same salary are counted together.

Now look at this problem

The table shows the number of workers of different categories in an office, grouped according to their daily wages:

Daily wage (Rupees)	Number of workers
675	8
730	4
755	4
780	3
840	1

What is the mean daily wage?

To compute this, we must find the number of workers and their total daily wage.

We can find the total of workers getting the same wage by multiplication. For example, the total daily wages of those getting daily wage of 675 rupees is

$$675 \times 8 = 5400 \text{ rupees}$$

Like this we can calculate the total wages of workers in each category separately:

Daily wage (rupees)	Number of workers	Total wage (Rupees)
675	8	5400
730	4	2920
755	4	3020
780	3	2340
840	1	840
Total	20	14520

Now can't we compute the mean wage?

$$\text{Mean wage} = 14520 \div 20 = 726 \text{ rupees}$$

In this problem, the minimum of the wages is 675 rupees and the maximum of the wages is 840 rupees. The mean wage is more than 675 rupees and less than 840 rupees. Is this true in all instances?

See this example

- Consider any 8 numbers with minimum 100 and maximum 200
- Their sum is more than $100 \times 8 = 800$ and less than $200 \times 8 = 1600$
- So, the mean got by dividing the sum by 8, is more than $800 \div 8 = 100$ and less than $1600 \div 8 = 200$

For numbers other than 8, 100 and 200 also we can reason in the same manner

Thus we have the general result:

The arithmetic mean of a set of numbers is between their minimum and maximum

(1) In a T20 match, 51 runs were scored in the first 5 overs

(i) What is the mean run rate then?

(ii) If this run rate is maintained, what is the total they can expect?

(2) The table below shows the children in a class grouped according to their marks in a math test:

Marks	Children
2	1
3	2
4	5
5	4
6	6
7	11
8	10
9	4
10	2

(i) What is the mean marks of the class?

(ii) How many got less marks than the mean?

(iii) How many got more marks than the mean?

(3) The details of rubber sheets a farmer got during a month are shown below:

Rubber (kg)	Days
9	3
10	4
11	3
12	3
13	5
14	6
16	6

(i) How many kilograms of rubber did he get a day on average in this month?

(ii) The price of a kilogram of rubber is 175 rupees. How much did he get a day on average this month from rubber?

(4) The table below shows the days in a month sorted according to the amount of rainfall in a locality:

Rainfall (mm)	Days
54	3
56	5
58	6
55	3
50	2
47	4
44	5
41	2

What is the mean rainfall per day during this month?

Frequency tables

We have seen in class 8 how data is tabulated as classes and frequencies when the amount of data is large. Let's see how we compute the arithmetic mean from such a table.

The table classifies 40 persons who took a test according to the marks they scored: Calculate the mean marks scored.

Marks	Persons
0 - 10	4
10 - 20	6
20 - 30	16
30 - 40	8
40 - 50	6

To calculate the mean, we must divide the total marks by the number of persons.

Here, how do we find the total marks?

From the first line of the table, we know only that 4 persons got marks between 0 and 10. Without knowing the exact marks each got, how do we compute the total? Same is the problem with the other rows.

In such cases, we will have to make some assumptions about the missing information. Even though we don't know the exact marks of the 4 persons listed in the first row, we know that it is between 0 and 10. So their mean marks is also between 0 and 10. (We have seen that the arithmetic mean of a set of numbers is between their minimum and maximum.)

Not only that, in most such cases, the mean is nearly half way between 0 and 10.

So in computing the mean from tables such as this, we proceed under the assumption that the mean of each is at the exact middle of the class. It is called the *class mark*.

For example in the table above, we assume that the mean marks of the 4 persons listed in the first row is 5, which is at the exact middle of 0 and 10; and then compute their total marks as 4 x 5 = 20

We can compute the total marks of persons in each class under this assumption and extend the table:

Marks	Persons	Class mark	Total marks
0 - 10	4	5	20
10 - 20	6	15	90
20 - 30	16	25	400
30 - 40	8	35	280
40 - 50	6	45	270
Total	40		1060

Now we can compute the mean marks of all the 40 persons together as

$$1060 \div 40 = 26.5$$

(1) The table below shows the children in a class, grouped according to their heights:

Height (cm)	Number of children
148 - 152	8
152 - 156	10
156 - 160	15
160 - 164	10
164 - 168	7

What is the mean height?

(2) The table below shows the classification of teachers in a university based on their ages:

Age	Number of teachers
25 - 30	6
30 - 35	14
35 - 40	18
40 - 45	20
45 - 50	5
50 - 55	4
55 - 60	3

Calculate the mean age of the teachers

(3) The classification of a group of children according to their weights is given in the table below:

Weight (kg)	21 - 23	23 - 25	25 - 27	27 - 29	29 - 31	31 - 33
Number of children	4	7	8	6	3	1

Calculate the mean weight.

Notes

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Notes

Notes

CONSTITUTION OF INDIA

Part IV A

FUNDAMENTAL DUTIES OF CITIZENS

ARTICLE 51 A

Fundamental Duties- It shall be the duty of every citizen of India:

(a) to abide by the Constitution and respect its ideals and institutions, the National Flag and the National Anthem;

(b) to cherish and follow the noble ideals which inspired our national struggle for freedom;

(c) to uphold and protect the sovereignty, unity and integrity of India;

(d) to defend the country and render national service when called upon to do so;

(e) to promote harmony and the spirit of common brotherhood amongst all the people of India transcending religious, linguistic and regional or sectional diversities; to renounce practices derogatory to the dignity of women;

(f) to value and preserve the rich heritage of our composite culture;

(g) to protect and improve the natural environment including forests, lakes, rivers, wild life and to have compassion for living creatures;

(h) to develop the scientific temper, humanism and the spirit of inquiry and reform;

(i) to safeguard public property and to abjure violence;

(j) to strive towards excellence in all spheres of individual and collective activity so that the nation constantly rises to higher levels of endeavour and achievements;

(k) who is a parent or guardian to provide opportunities for education to his child or, as the case may be, ward between age of six and fourteen years.

CHILDREN'S RIGHTS

Dear Children,

Wouldn't you like to know about your rights? Awareness about your rights will inspire and motivate you to ensure your protection and participation, thereby making social justice a reality. You may know that a commission for child rights is functioning in our state called the ***Kerala State Commission for Protection of Child Rights.***

Let's see what your rights are:

- Right to freedom of speech and expression.
- Right to life and liberty.
- Right to maximum survival and development.
- Right to be respected and accepted regardless of caste, creed and colour.
- Right to protection and care against physical, mental and sexual abuse.
- Right to participation.
- Protection from child labour and hazardous work.
- Protection against child marriage.
- Right to know one's culture and live accordingly.
- Protection against neglect.
- Right to free and compulsory education.
- Right to learn, rest and leisure.
- Right to parental and societal care, and protection.

Major Responsibilities

- Protect school and public facilities.
- Observe punctuality in learning and activities of the school.
- Accept and respect school authorities, teachers, parents and fellow students.
- Readiness to accept and respect others regardless of caste, creed or colour.

Contact Address:

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'Sree Ganesh', T. C. 14/2036, Vanross Junction
Kerala University P. O., Thiruvananthapuram - 34, Phone : 0471 - 2326603
Email: childrights.cpcr@kerala.gov.in, rte.cpcr@kerala.gov.in
Website : www.kescpcr.kerala.gov.in

Child Helpline - 1098, Crime Stopper - 1090, Nirbhaya - 1800 425 1400

Kerala Police Helpline - 0471 - 3243000/44000/45000

Online ***R. T. E Monitoring : www.nireekshana.org.in***