MATHEMATICS

PART - 2

Standard

X

Government of Kerala
Department of General Education

Prepared by
State Council of Educational Research and Training (SCERT) Kerala

2025

THE NATIONAL ANTHEM

Jana-gana-mana adhinayaka, jaya he
Bharatha-bhagya-vidhata
Punjab-Sindh-Gujarat-Maratha
Dravida-Utkala-Banga
Vindhya-Himachala-Yamuna-Ganga
Uchchala-Jaladhi-taranga
Tava subha name jage,
Tava subha asisa mage,
Gahe tava jaya gatha
Jana-gana-mangala-dayaka jaya he
Bharatha-bhagya-vidhata
Jaya he, jaya he, jaya he,
Jaya jaya jaya, jaya he.

PLEDGE

India is my country. All Indians are my brothers and sisters.

I love my country, and I am proud of its rich and varied heritage. I shall always strive to be worthy of it.

I shall give my parents, teachers and all elders, respect and treat everyone with courtesy.

To my country and my people, I pledge my devotion. In their well-being and prosperity alone, lies my happiness.

MATHEMATICS

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State Council of Educational Research and Training (SCERT)
Poojappura, Thiruvananthapuram 695012, Kerala
Website : www.scertkerala.gov.in
e-mail : scertkerala@gmail.com, Phone : 0471 - 2341883
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Dear children,

We have seen in previous classes how measurements are converted to numbers and how problems about measurements are solved by converting them to number problems and then to algebraic problems. Thus we came to understand how relations between operations on pure numbers develop into algebraic principles, which in turn are used to solve problems about measurements. We also saw how special properties of various figures are turned into geometrical theorems.

In this textbook, we move a little more into these various avenues of mathematics. Here we get the first glimpses of how the usually separate disciplines of algebra and geometry merge into a new branch of mathematics. The basic ideas of probability theory, which is used in all sciences and which is an essential component of artificial intelligence used widely now.

The fundamental ideas needed for those who opt for mathematics as a part of their continuing education are all discussed in our textbooks. The inherent value of these books is the culture of mathematical thinking which everyone should be able to acquire as they finish one stage of their basic education. The essence of this culture is the ability to analyze everything logically and form conclusions based only on such reflection

With love and regards,

Dr. Jayaprakash R.K.
Director
SCERT Kerala

TEXTBOOK DEVELOPMENT TEAM

Advisor

Dr. E. Krishnan
Head (Retd.), Department of Mathematics,
University College, Thiruvananthapuram

Chairperson

Dr. P. Ramesh Kumar
Head (Retd.), Department of Mathematics,
University of Kerala
Karyavattom, Thiruvananthapuram

Experts

Dr. Shiki M.
Professor,
College of Engineering Trivandrum,
Thiruvananthapuram

Sumesh S.S
Assistant Professor of Mathematics,
Department of Mathematics and Statistics,
Mar Ivanious College (Autonomous),
Thiruvananthapuram

Members

R. Ramanujam
HSST Mathematics,
M.N.K.M. Govt. Higher Secondary
School, Pulapatta, Palakkad

Vijayakumaran T.K.
HSST (Retd.),
G.M.R.H.S.S. for Girls,
Pravanadukkam, Kasaragod

C.T. Firoz Babu
HST Mathematics,
GGVHSS Wandoor, Malappuram

Unnikrishnan M.V.
HST Mathematics,
GHSS, Kadannappally P.O,
Kadannappally, Kannur

Nandakumar C.
HST Mathematics,
Chothavoor Higher Secondary School,
Champad, Kannur

Sreejith Muriyampath
HST Mathematics, GHSS Madappally,
Vadakara, Kozhikode

Padmaprasad K.
HST Mathematics, Pandallur HSS,
Kadambode Post, Malappuram

Ubaidulla K.C.
HST Mathematics,
SOHS Areekode, Malappuram

Firoza Moidu
HST Mathematics, Calicut Girls VHSS,
Kundungal, Kallayi, Kozhikode

Priya Raj
HST Mathematics,
MGD High School, Puthusserry,
Puthusserry South P.O,
Pathanamthitta

Ayishakutty K.K.
HST Mathematics,
GHS, Beenachi,
Sulthanbethery, Wayanad

Rajesh M.
HST Mathematics,
Kalladi HSS Kumaramputhur,
Mannarkkad, Palakkad

Ullas U.G., (Artist)
HST Physical Science, CBHSS Vallikunnu, Malappuram

Academic Coordinator

Dr. Anilkumar A.K.
Research Officer, SCERT

State Council of Educational Research and Training (SCERT) Kerala
Vidhyabhavan, Poojappura, Thiruvananthapuram 695 012

CONTENTS

Certain icons are used in this textbook for convenience


Let's do problems


Project


ICT possibilities

THE CONSTITUTION OF INDIA

PREAMBLE

WE, THE PEOPLE OF INDIA, having solemnly resolved to constitute India into a [1]**[SOVEREIGN SOCIALIST SECULAR DEMOCRATIC REPUBLIC]** and to secure to all its citizens :

JUSTICE, social, economic and political;

LIBERTY of thought, expression, belief, faith and worship;

EQUALITY of status and of opportunity; and to promote among them all

FRATERNITY assuring the dignity of the individual and the [2][unity and integrity of the Nation];

IN OUR CONSTITUENT ASSEMBLY this twenty-sixth day of November, 1949 do **HEREBY ADOPT, ENACT AND GIVE TO OURSELVES THIS CONSTITUTION.**

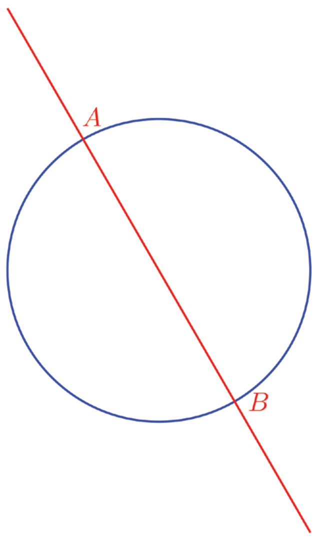
---

1. Subs. by the Constitution (Forty-second Amendment) Act, 1976, Sec.2, for "Sovereign Democratic Republic" (w.e.f. 3.1.1977)
2. Subs. by the Constitution (Forty-second Amendment) Act, 1976, Sec.2, for "Unity of the Nation" (w.e.f. 3.1.1977)

8 TANGENTS

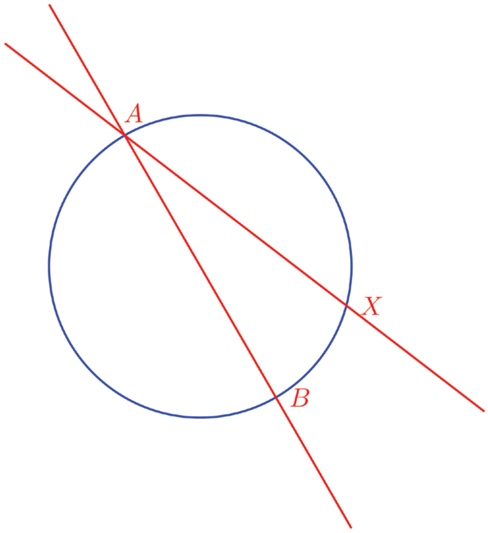
Lines and Circles

See this picture:

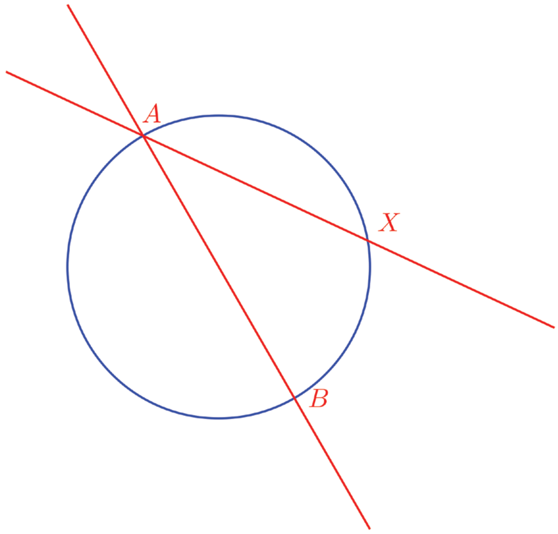


AB is the diameter of the circle through the point A on it; and it is extended a bit to either side.

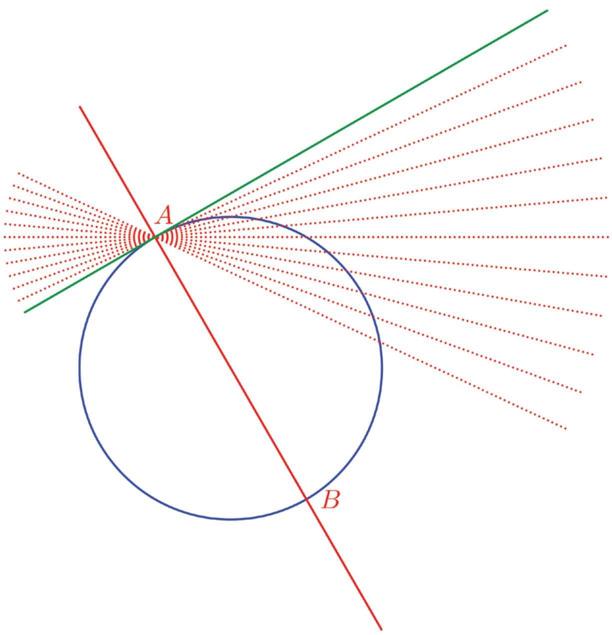
This picture shows the chord AX through another point X on the circle, extended as before:



What if we keep A fixed and move the point X a little closer to A?



Suppose we keep on moving X closer and closer to A along the circle?



In GeoGebra, draw a circle centred at a point O and mark two points A and X on it. Draw the lines OA and AX. What happens to the line AX as the position of X is brought closer and closer to A? And when the point X reaches A? Join OX. See what happens to the angles OAX and AOX when X is brought close to A.

The line AX gets closer and closer to the green line in the picture, doesn't it?

What can we say about the green line?

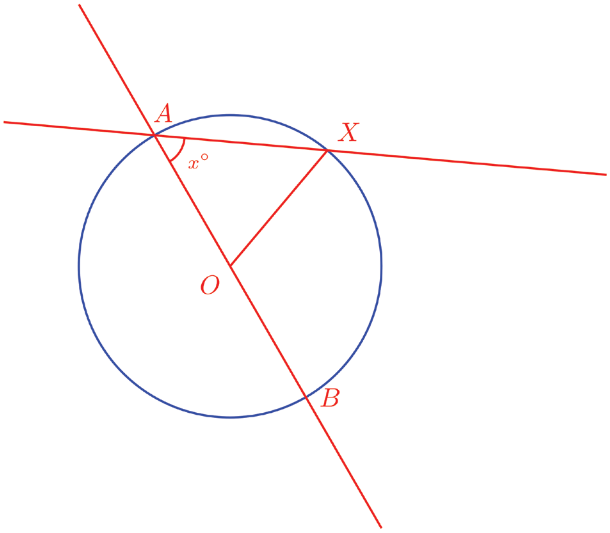
It just touches the circle at A, right?

This line is called the **tangent** to the circle at the point A.

Look at the picture again; see any relation between the diameter and the tangent?

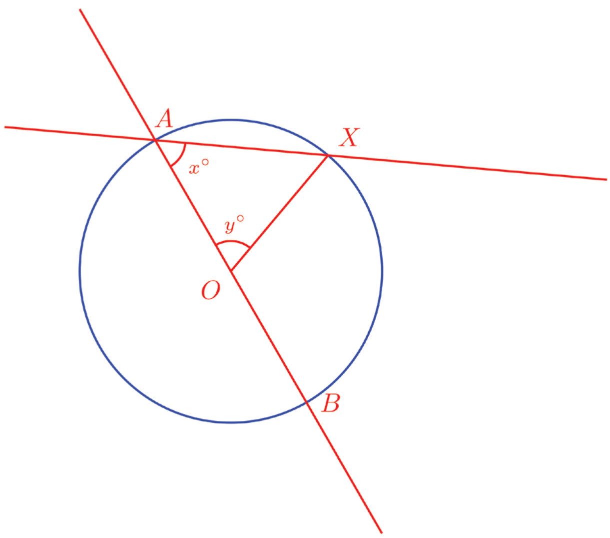
Doesn’t the tangent seem to be perpendicular to the diameter?

To check this, let’s inspect what happens to the angle between the diameter AB and the chord AX as X is moved closer and closer to A:



As X is moved closer to A, this angle, marked $x°$ in the picture, becomes larger. How large can it become?

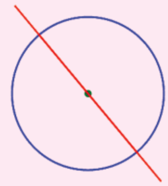
To check it, let’s take the central angle of the chord AX as $y°$:



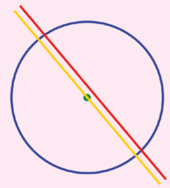
As X is moved closer to A, the length of the chord AX and its central angle become smaller; in other words, the number y gets closer to zero.

Sliding lines

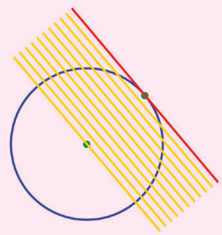
See this picture:



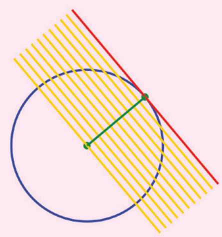
A circle and a line through its centre. Suppose we slide the line up a bit:



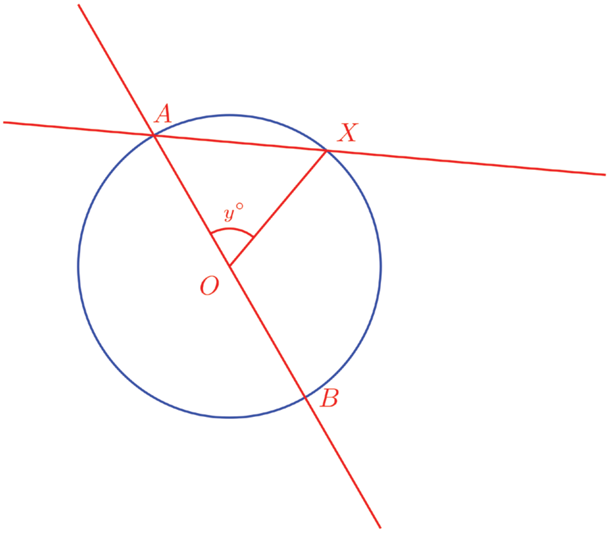
As we go on sliding the line slowly, we get a line which passes through a *single* point of the circle, right?

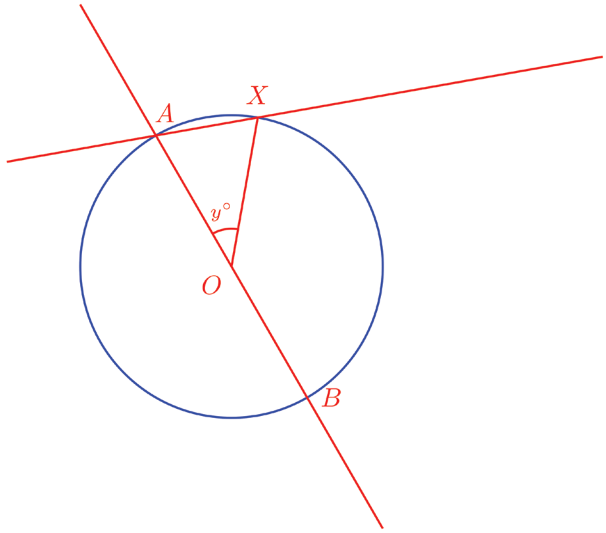


And the line joining the centre and this final point is perpendicular to all these lines:



In GeoGebra, draw a circle and its radius. Mark a point on the radius (Point on Object) and draw the perpendicular through this. Change the position of the point. When it reaches the circle, what happens to the perpendicular?

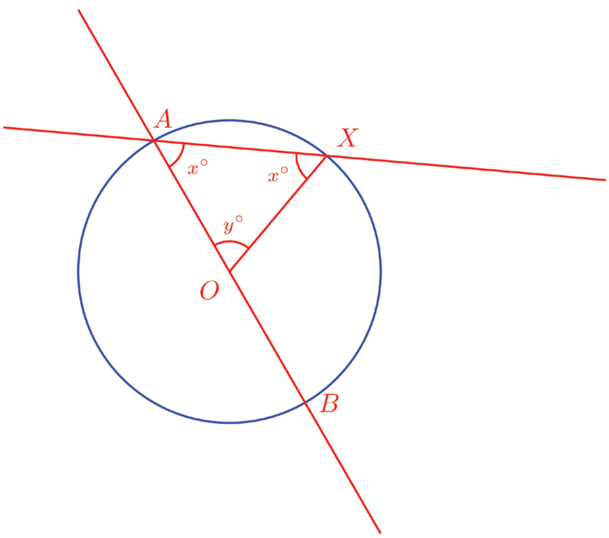


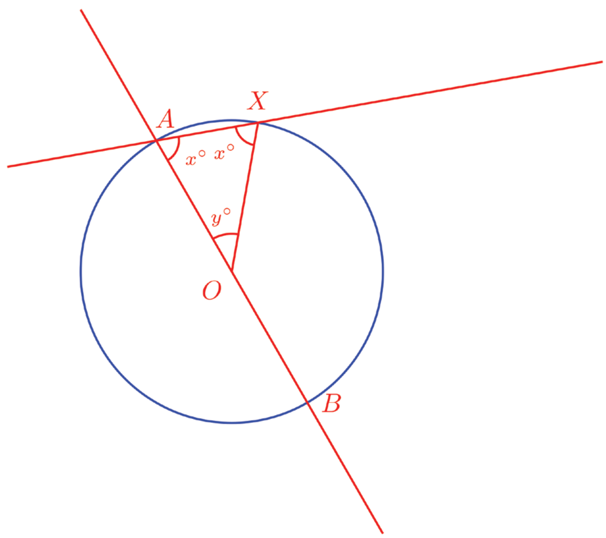


What we want to know is, what happens to the number x; what is the relation between x and y?

The sides OA and OX of triangle OAX are radii of the circle and so are equal. So the angles opposite to these sides in the triangle are also equal (The section **Isosceles triangles** of the lesson, **Equal Triangles** in the Class 8 textbook).

So, the third angle of triangle OAX is also $x°$:





Since the sum of the angles of a triangle is 180°, we have

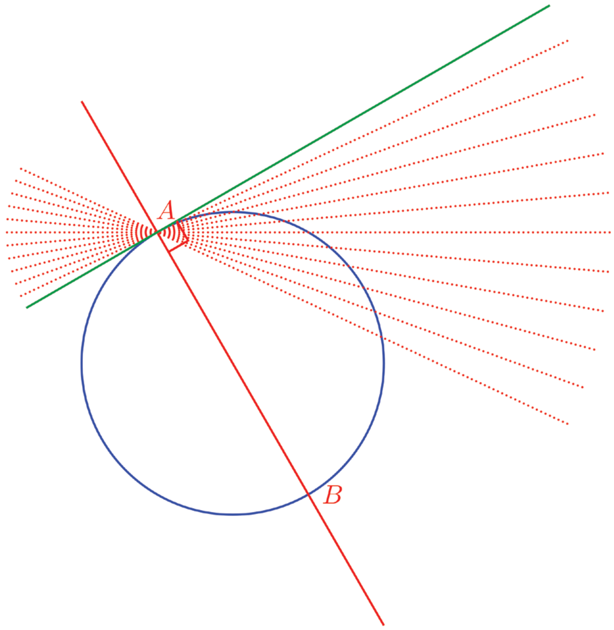
$$2x + y = 180$$

This gives

$$x = \frac{1}{2}(180 - y) = 90 - \frac{1}{2}y$$

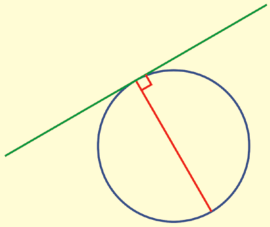
From this, we can see that as the number y gets closer to zero, the number x gets closer to 90°.

Thus as the point X is moved closer to A, the angle between the extended chord AX and the diameter AB gets closer to 90°; and when it is the tangent at A, the angle is exactly 90°:



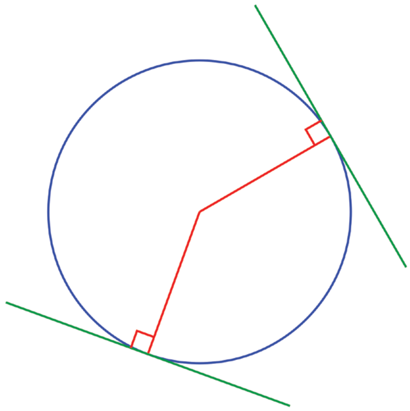
We state this as a general result:

The tangent at a point to a circle is perpendicular to the diameter through that point

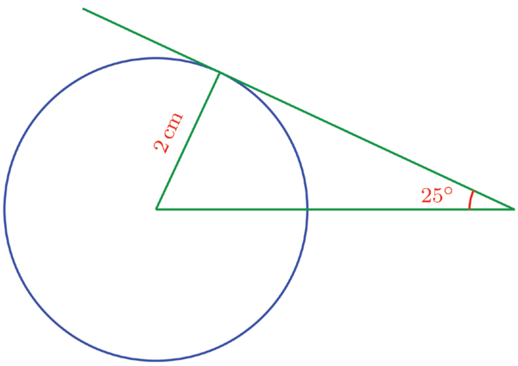


Using this, we can draw the tangent at any point on a circle; we just need to draw the perpendicular through this point to the diameter through this point.

Instead of drawing the whole diameter, we need only draw the radius through the point.



Now see this picture:



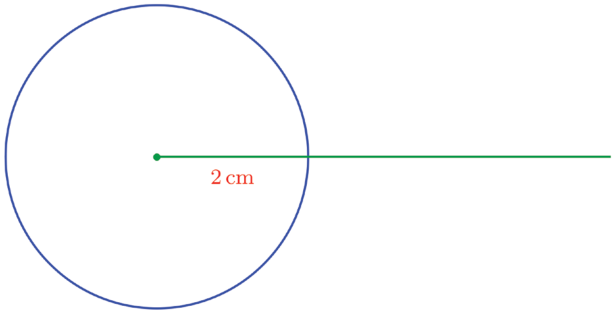
To draw a tangent to a circle in GeoGebra, we need only select Tangents and click on the circle and the point of contact. If the point is on the circle, we get the tangent at that point. What if the point is outside the circle?

Draw the tangent through a point on the circle and set Trace On for it; and then try Animation for the point of contact.

The top line is a tangent to the circle.

Can you draw this picture in your notebook?

First we draw a circle of radius 2 centimetres, and the horizontal line through the centre:

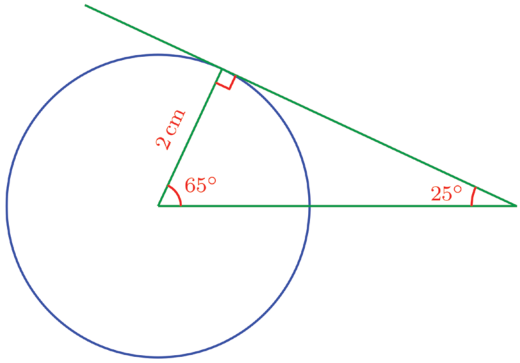


Now at which point on the circle should we draw the tangent?

It should meet the first line at an inclination of 25°.

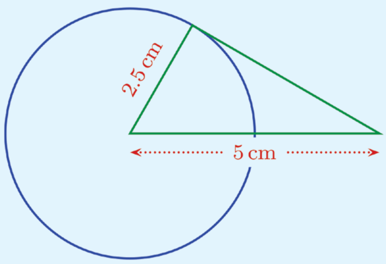
Look at the first picture again.

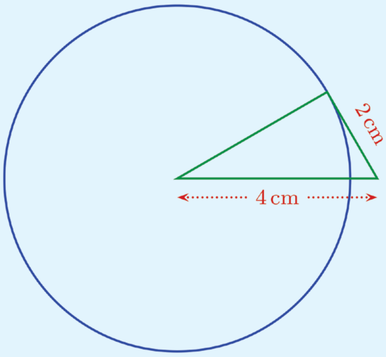
The top angle of the triangle in the picture is 90°; and another angle is 25°. So what is the third angle?



Now can't you complete the picture?

(1) In each of the two pictures below, a tangent to the circle, the radius through the point of contact and another line through the centre are drawn to make a triangle:





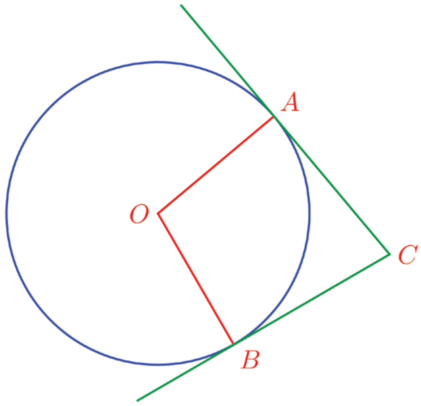
Draw these in your notebook.

(2) Prove that the tangents drawn at the two ends of a diameter of a circle are parallel.

(3) If the tangents are drawn to a circle at the ends of a pair of perpendicular diameters, what kind of quadrilateral do they form?

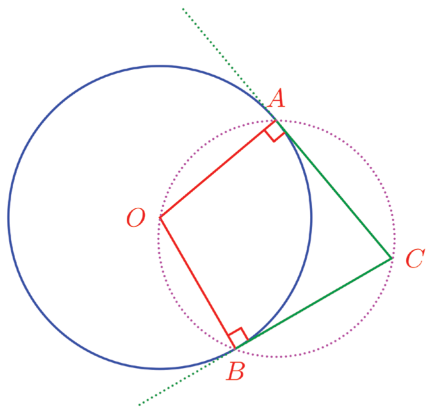
Tangents and angles

See this picture:



Tangents are drawn at the points A and B on a circle centred at O, and they meet at the point C.

In the quadrilateral $OACB$, the angles at the opposite corners A and B are right angles and so their sum is 180°. This quadrilateral is therefore cyclic:

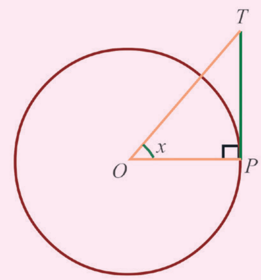


Name and meaning

The word *tangent* comes from the Latin root *tangere*, meaning *to touch*.

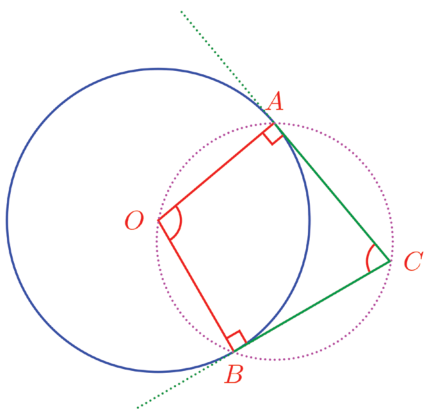
The tan measure used in trigonometry is also an abbreviation of the word *tangent*, right? What is the connection between this measure of an angle and a line touching a circle?

See this picture:



If we take the radius of the circle as 1, then the length of the tangent PT is indeed tan x, isn't it?

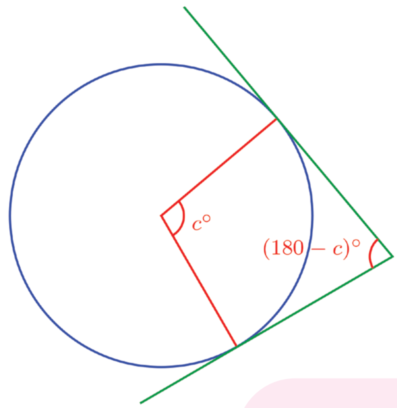
In such a quadrilateral, the sum of the other two angles is also 180°.



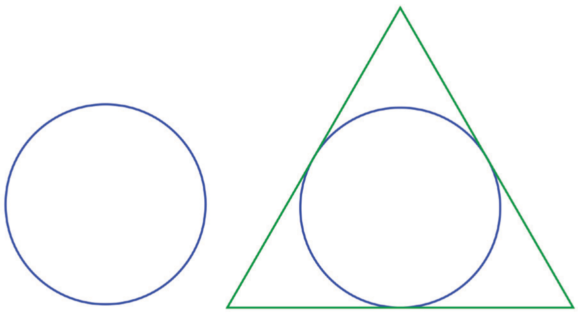
This we can state as a general result:

The angle at the centre of a circle between two radii, and the angle between the tangents at the ends of radii add up to 180°

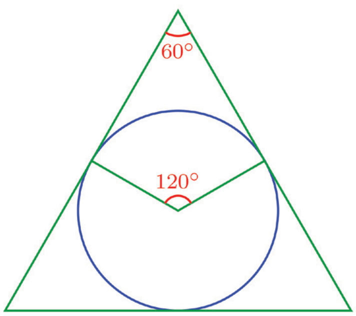
Draw a circle centred at O in GeoGebra and mark points A and B on it. Draw tangents at these points and mark their point of intersection as C. Draw the quadrilateral OACB. Is it cyclic? Using Circle through Three Points, we can draw the circle passing through O, A, B. Change the positions of A and B. What happens to C, when they get closer and closer? And when they move farther and farther away? What happens when they are the end points of a diameter?



Let's look at a figure, we can draw using this; an equilateral triangle just covering a circle:



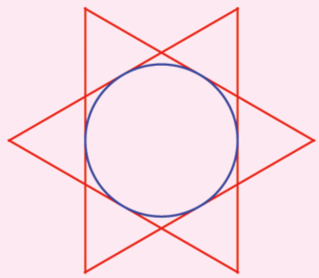
The sides of the triangle are tangents to the circle. Since the triangle is equilateral, the angle between each pair of them is 60°. What about the angles between two radii through the points of contact?



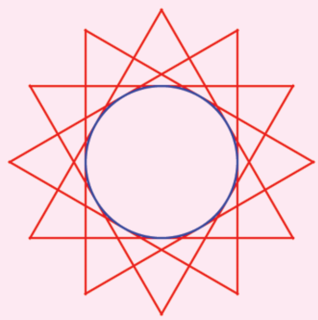
Like this, we can see that the angles between the radii are all 120°:

Circle from lines

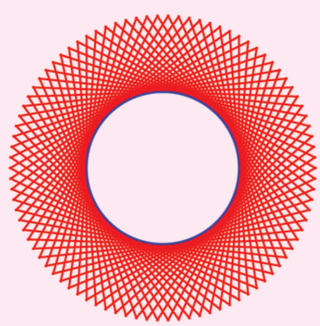
See this picture of a star made by six tangents to a circle:

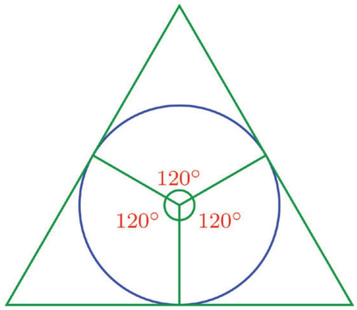


We can increase the number of tangents to 12:

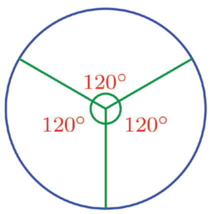


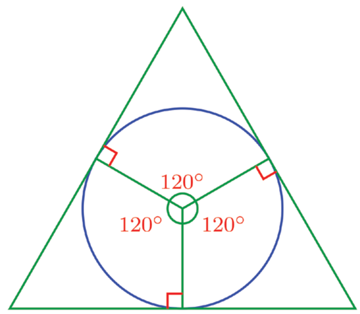
And this is a picture drawn by a computer, using 90 tangents:





So, if we draw three radii separated by 120° and draw perpendiculars through their ends, we get the tangents; and thus the triangle we want:

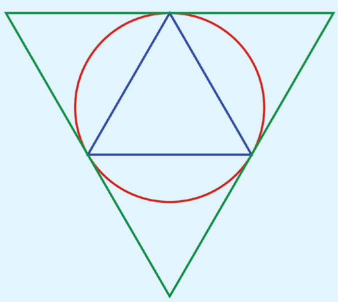




Draw a circle of radius 3 centimetres and draw a triangle like this.

(1) Draw a circle of radius 2.5 centimetres. Draw a triangle of angles 40°, 60°, 80°, with its sides touching the circle.

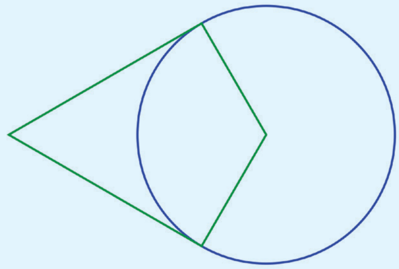
(2) In the picture, the small (blue) triangle is equilateral. The sides of the larger (green) triangle are tangents at the vertices of the smaller triangle to its circumcircle.



(i) Prove that the larger triangle is also equilateral and its sides are double those of the smaller triangle.

(ii) Draw this picture with the sides of the smaller triangle as 3 centimetres.

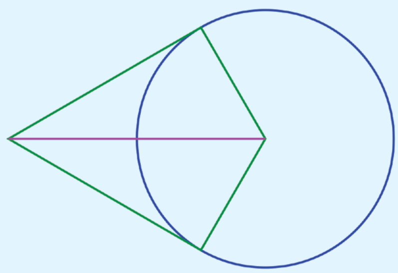
(3) The picture shows two tangents to a circle and the radii through the points of contact:



(i) Prove that the lengths of the tangents are equal.

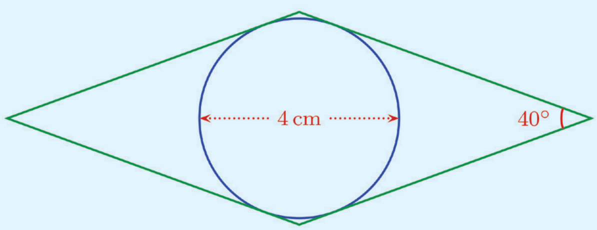
(ii) Prove that the line joining the centre of the circle to the point of intersection of the tangents bisects the angle between the radii.

Draw a circle in GeoGebra and mark a point on it. Select Angle with Given Size and click on this point and the centre of the circle; in the dialogue window, type 120° as the Angle, we get another point on the circle. Again click on this point and at the centre of the circle, then type 120° to get another point on the circle. Draw tangents to the circle through these three points and mark their points of intersection. Use Segment to join these points to get a triangle. Now we can hide the tangents. Set Trace On for the sides of the triangle and Set Animation On for the first point chosen. What happens?



(iii) Prove that this line is the perpendicular bisector of the chord joining the points of contact.

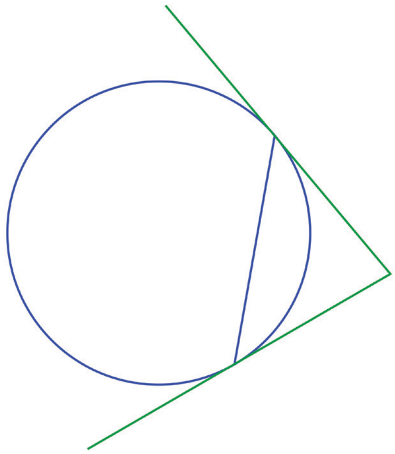
(4) The picture shows a rhombus with each side tangent to a circle within it:



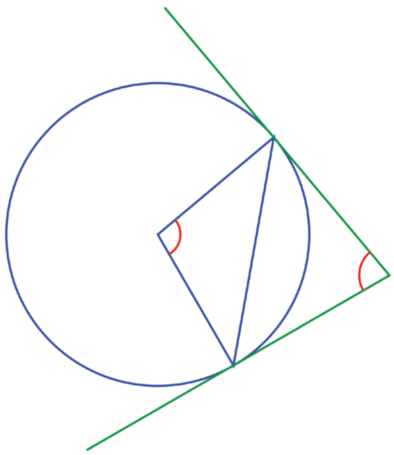
Draw this in your notebook.

Chord and tangent

The picture shows the tangents to a circle at the two ends of a chord:

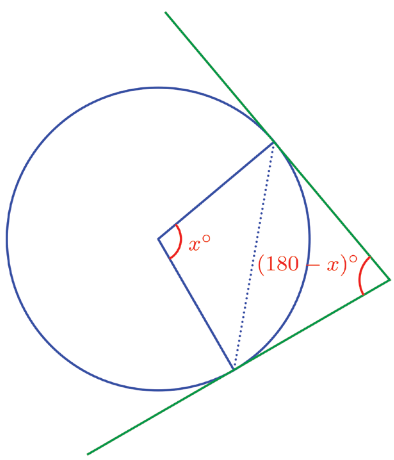


What is the relation between the central angle of the chord and the angle between the tangents?



Draw a circle centred on a point A in GeoGebra and mark points B and C on it. Draw tangents at these points and mark their point of intersection D. Draw the chord BC. Join AB and AC. What is the relation between the angles BAC and BDC? Change the positions of B and C and see.

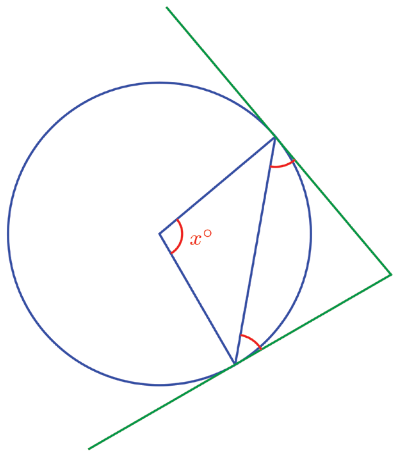
As seen before, their sum is 180°.



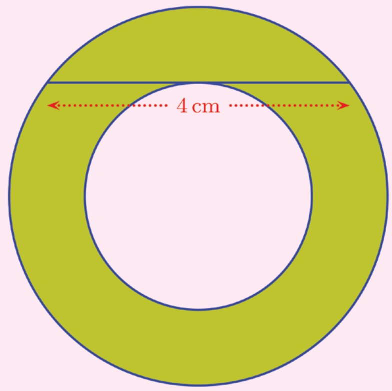
So, we can state the earlier result like this:

> The angle between the tangents at the ends of a chord in a circle is the central angle of the chord subtracted from 180°

Next let's see the relation between the central angle of the chord and the angles which the tangents make with the chord itself:

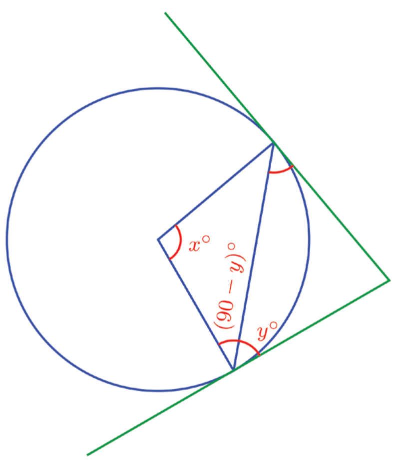


Area problem



What is the area of the green ring in the picture?

Let's take the angle which the lower tangent makes with the chord as $y°$. Since the angle between the tangent and the radius through the point of contact is 90°, we can mark the angles like this:



> Draw a circle in GeoGebra and a chord. Draw tangents at the ends of this chord. Mark the central angle of the chord and one of the angles between the chord and a tangent. What is the relation between these angles? Draw different chords and check.

In the (blue) triangle inside the circle, two of the sides are equal, being radii of the circle; so the angles of the triangle opposite these sides are also equal.

Thus the third angle of the triangle is also $(90 - y)°$. Since the sum of the angles of a triangle is 180°, we get

$$x + (90 - y) + (90 - y) = 180$$

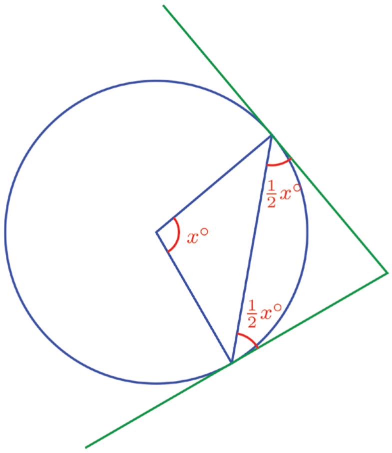
From this, we get

$$x - 2y = 0$$

This gives

$$y = \frac{1}{2}x$$

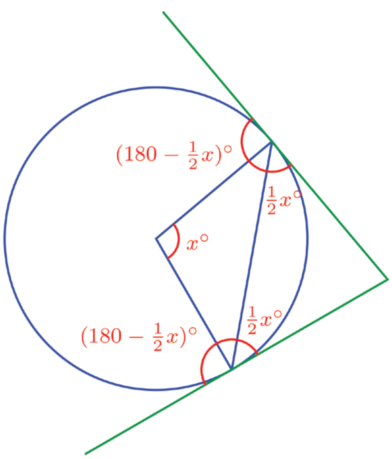
In the same way, the angle which the upper tangent makes with the chord can also be seen to be $\frac{1}{2}x^\circ$:



Thus we get this result:

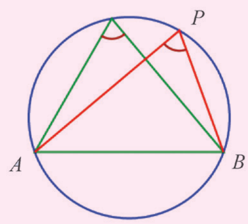
> The angles which the tangents at the ends of a chord of the circle make on one side of the chord are both equal to half the central angle of the chord.

What about the angles made on the other side of the chord?

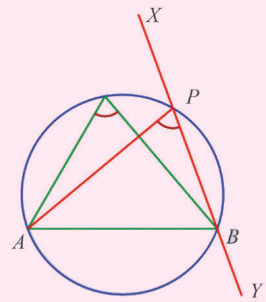


Unchanging angle

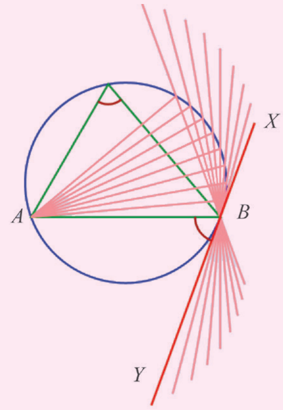
We have seen that all angles in the same segment of a circle are equal.



Let's extend *PB* to both sides:

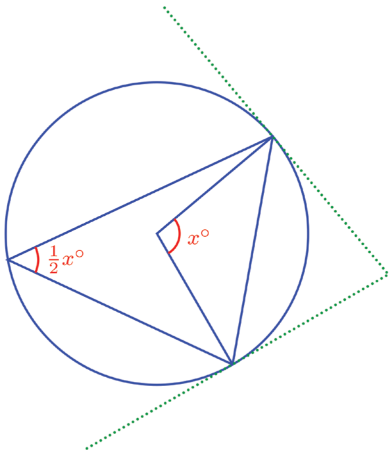


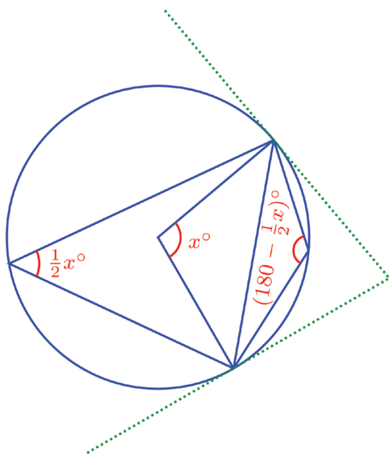
Now what happens as we move *P* along the circle towards *B* and finally to coincide with *B*?



The line *XY* becomes the tangent at *B*; and the angle is not changed.

Here we recall a result seen in the chapter **Circles and Angles**. If the central angle of an arc is $x°$, then its ends make an angle $\frac{1}{2}x^\circ$ on the opposite arc and an angle $\left(180-\frac{1}{2}x\right)^\circ$ on the same arc:



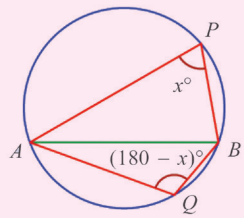


In other words, here the angles which the tangents make on the right side of the chord is $\frac{1}{2}x^\circ$ and any angle in the segment on the left of the chord is also $\frac{1}{2}x^\circ$.

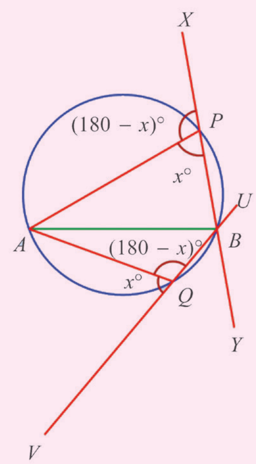
What about the angle $\left(180-\frac{1}{2}x\right)^\circ$ which the tangents make on the left side of the chord?

Flip-flop angles

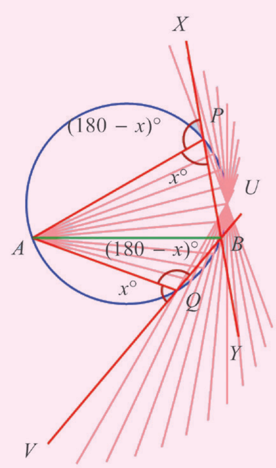
We have seen that the sum of the angles in alternate segments of a circle is 180°.



Let's extend the lines as before:



Suppose P moves along the circle to Q:

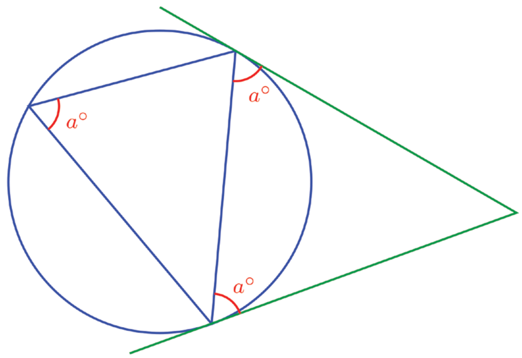


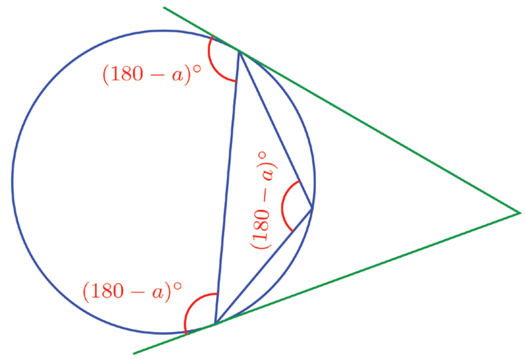
The angle below AP is $x°$ and the angle above AP is $(180 – x)°$. And this is so throughout the motion.

So in general, we can say this about the angles which tangents at the ends of a chord make with the chord:

> The angles which the tangents to a circle at the ends of a chord make on one side of the chord are both equal to the angle in the alternate segment

> Draw a circle with centre A and mark points B and C on it. Draw the tangents at these points and also the chord BC. Mark the angles which the tangents make on one side of the chord and the central angle of the chord. What is the relation between these angles? Mark a point D on the other side of the chord. Join BD and CD and mark the angles at D. What is the relation between the angle which the tangents make on one side of the chord and the angle D on the segment on the other side?

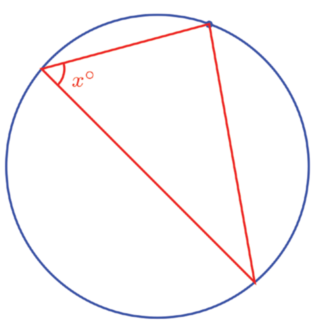


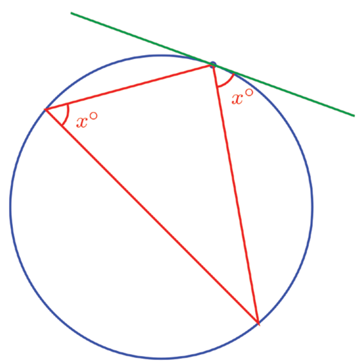


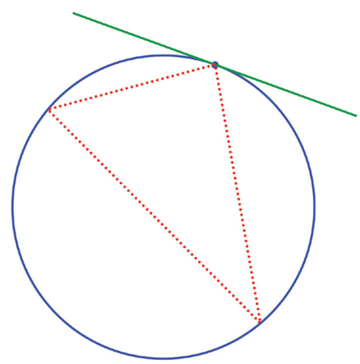
We usually draw the tangent to a circle at a point as the perpendicular to the radius through that point. We can use the above result to draw a tangent even when we do not know the centre.

For this, we need only draw a chord through the point, measure the angle which it makes on a point on the circle and draw the same angle on the other side of the chord:

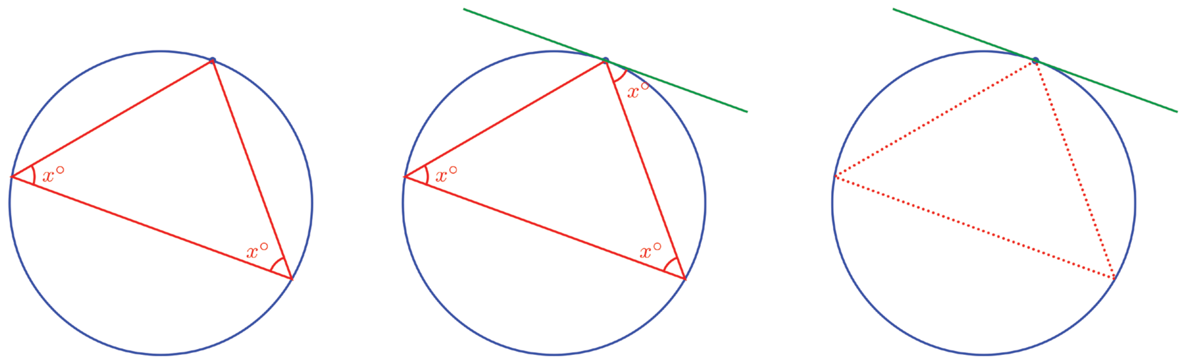
> Draw a circle in GeoGebra and draw tangents at two points A and B on it. Mark an angle between a tangent and the chord AB. Mark a point C on the circle and mark the angle ACB. What is the relation between the angles? Change the positions of A, B, C and check.



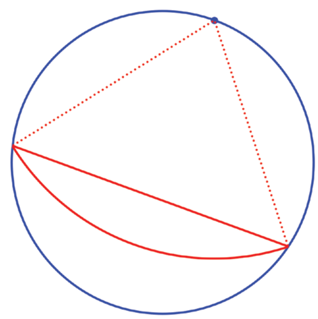




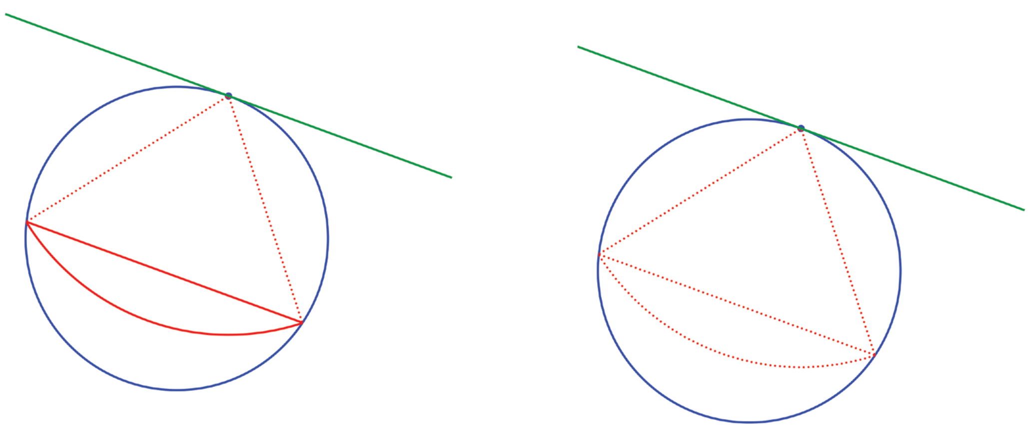
If the triangle in this picture is drawn isosceles, then we need only draw a line parallel to the base:



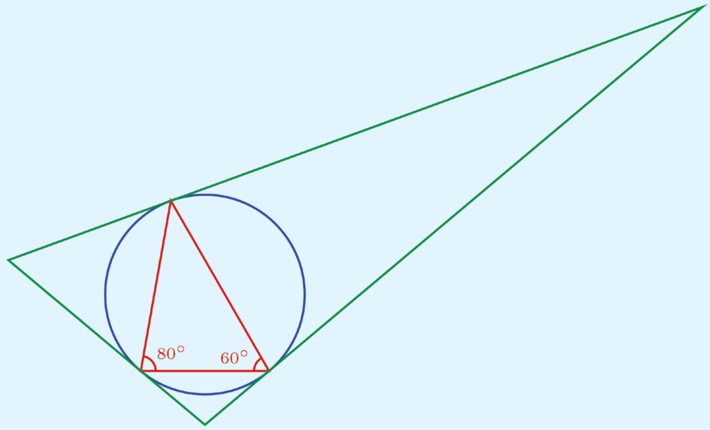
So to draw the tangent at a point on a circle, first draw an arc centred at this point and join the points at which it cuts the circle:



Now we need only draw a line parallel to this through the point to get the tangent:

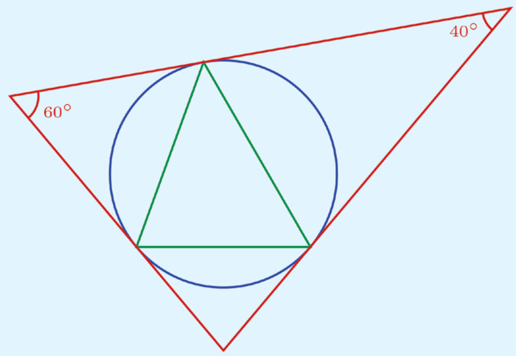


(1) In the picture, the sides of the larger triangle are the tangents to the circumcircle of the smaller triangle at its vertices:



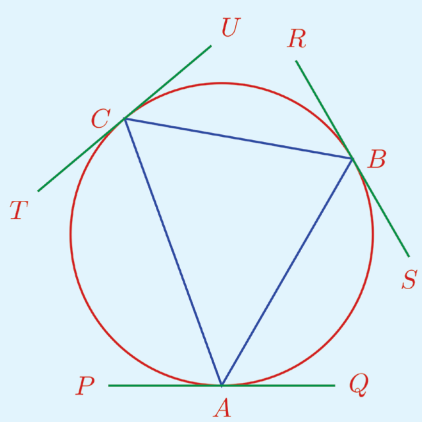
Calculate the angles of the larger triangle.

(2) In the picture, the sides of the larger triangle are tangents to the circle. Their points of contact are the vertices of the smaller triangle:



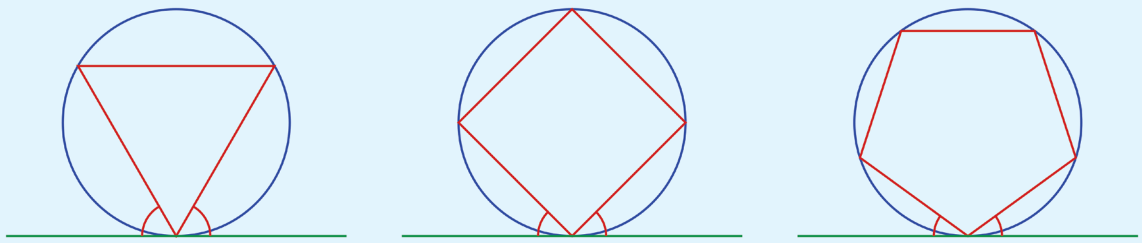
Calculate the angles of the smaller triangle.

(3) In the picture, *PQ*, *RS*, *TU* are tangents to the circumcircle of triangle *ABC* at its vertices.



Sort out the equal angles in the picture.

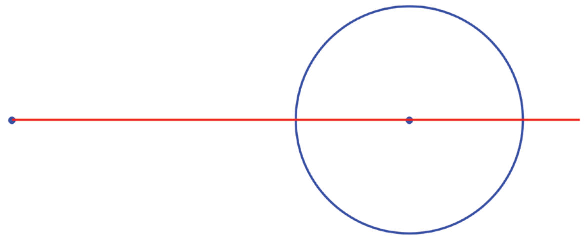
(4) In each picture below, a tangent is drawn to the circumcircle of a regular polygon, at a vertex:



In each picture, calculate the angles between the tangent and the sides of the polygon through the point of contact.

Tangent from outside

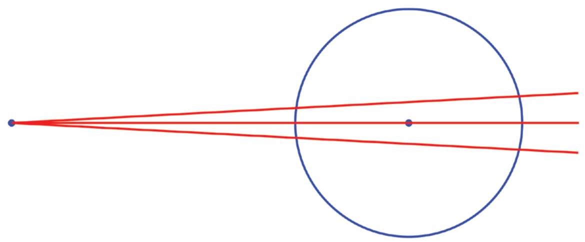
See this picture:



The centre of a circle is joined to a point outside the circle and the line is extended.

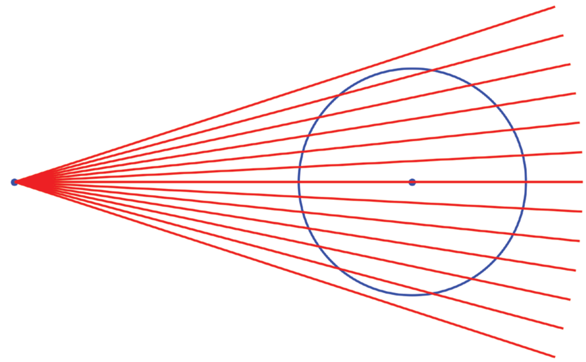
It cuts the circle at two points, which are the ends of a diameter of the circle.

What if we join the point outside the circle to a point inside the circle, which is slightly above or below the centre?



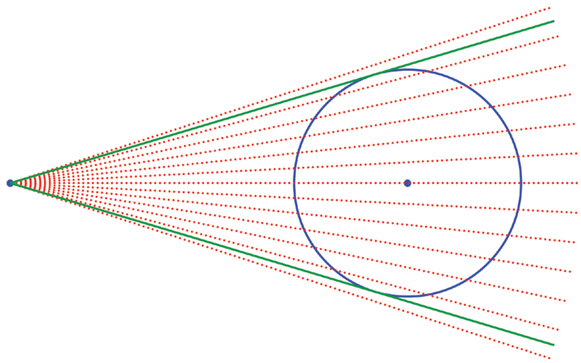
The points of intersection of the line with the circle get a bit closer to each other.

What if we continue like this?



The lines which cut the circle at points closer and closer to each other, have no intersection with the circle after a stage.

But among these, aren't there two lines, one below and one above, which just touch the circle?

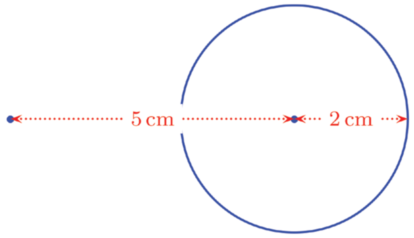


What do we see from this?

From a point outside a circle, two tangents can be drawn to the circle

But this doesn't say *how* we can draw such tangents.

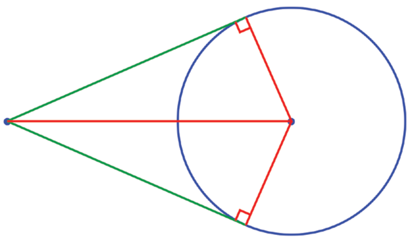
See this picture:



A point is marked 5 centimetres away from the centre of a circle of radius 2 centimetres.

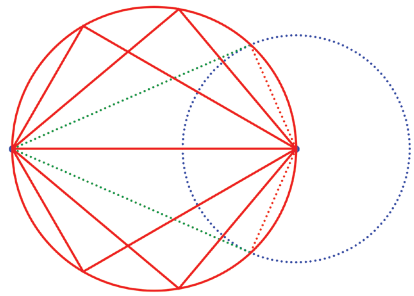
How do we draw two tangents from this point to the circle?

If we imagine how the figure would look like after drawing these tangents, we may perhaps get an idea as to how we can draw them. Recall that the tangents are perpendicular to the radii through the points of contact:



So what we want are pairs of perpendicular lines, through the centre of the circle and the point outside.

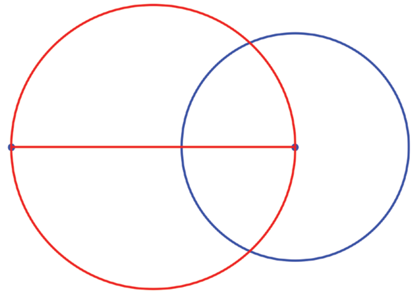
To draw a pair of such lines, we need only join these two points to any point on the circle with the line joining them as diameter, right?



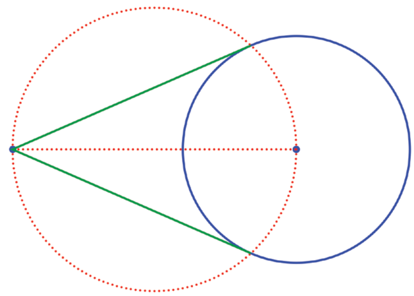
(Recall that angles in a semicircle are right angles, as seen in the chapter **Circles and Angles**).

We also must have one of the perpendicular lines to be the radius of the first circle. Thus the lines must intersect on the first circle. For this, we need only draw the lines through the points of intersection of the two circles, right?

Now we can draw the tangents. First join the centre of the circle and the point outside and draw the circle with this as the diameter:



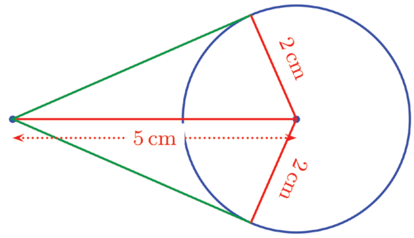
If we join the points of intersection of the circles with the point outside, we get our tangents:



(1) Draw a circle of radius 2.5 centimetres and draw tangents to it from a point 7 centimetres from the centre.

(2) Draw a circle of radius 3 centimetres and draw tangents to it from a point 7.5 centimetres from the centre.

In the problem we discussed, the radius of the circle is 2 centimetres and the distance of the point from the centre is 5 centimetres:



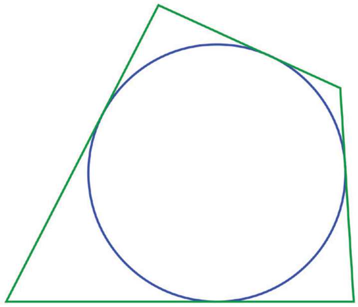
So using Pythagoras Theorem, we can calculate the lengths of the tangents (from the point outside to the point of contact) as:

$$\sqrt{5^2 - 2^2} = \sqrt{21} \text{ cm}$$

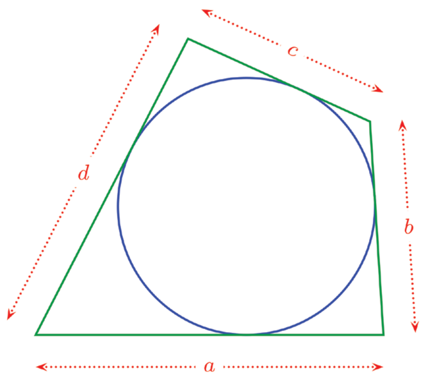
We have seen that if we draw two tangents to a circle, then the lengths of each from the point of contact to the point of their intersection are the same (third question of the section **Tangents and Angles**). This we can also state like this:

The two tangents from a point outside a circle have the same length

Let's look at a problem using this idea. The figure below shows the quadrilateral formed by the tangents to a circle at four points:

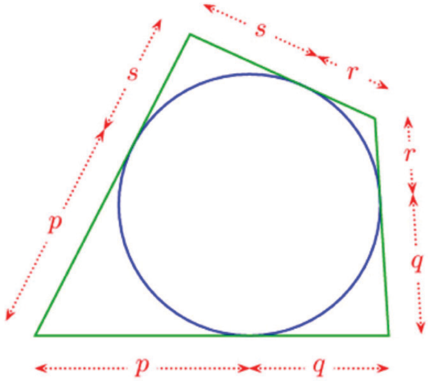


Let's take the lengths of the sides of this quadrilateral as *a, b, c, d* as shown below:



Draw a circle in GeoGebra and mark four points on it. Draw tangents at these points and mark their points of intersection. Draw the quadrilateral with these points as vertices. Now hide the tangents. Mark the lengths of the sides of the quadrilateral and note the relation between them. Change the positions of the points and check.

If we take the lengths of the tangents from each vertex as *p, q, r, s* then we can mark these lengths as shown below:



From these pictures, we get the relations between the lengths of the sides of the quadrilateral and the lengths of the tangents:

$$a = p + q$$
$$b = q + r$$
$$c = r + s$$
$$d = s + p$$

Do you see any relation between *a, b, c, d* from these?

$$a + c = p + q + r + s$$

$$b + d = q + r + s + p$$

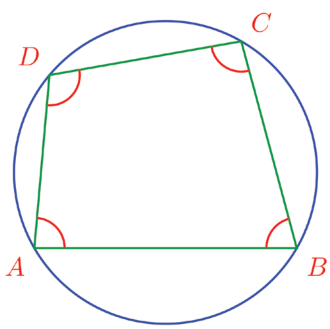
Thus we see that

$$a + c = b + d$$

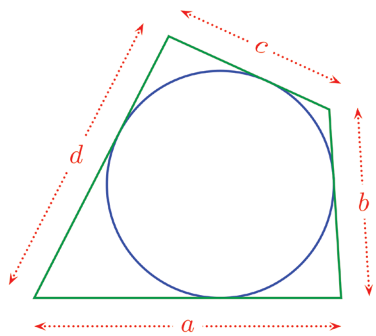
So what can we say about the lengths of the sides of the quadrilateral?

> In a quadrilateral formed by the tangents at four points on a circle, the sum of the lengths of opposite sides is equal

We have seen in the chapter **Circles and Angles** that in a quadrilateral formed by joining four points on a circle, the sum of each pair of opposite angles is 180°; that is in such a quadrilateral, the sum of the opposite angles is equal:

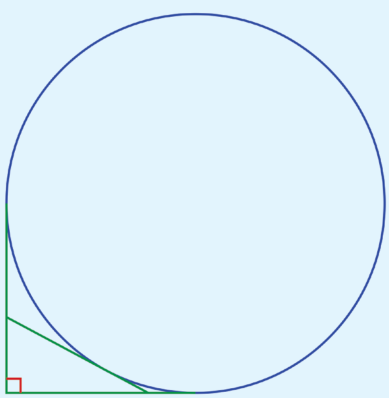


$\angle A + \angle C = \angle B + \angle D$



$a + c = b + d$

(1) In the figure, a triangle is formed by two perpendicular tangents and a third tangent to circle:

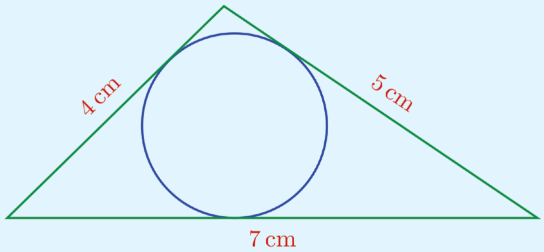


(i) Prove that the perimeter of the triangle is the sum of the lengths of the perpendicular tangents.

(ii) Prove that the length of each of the perpendicular tangents is equal to the radius of the circle.

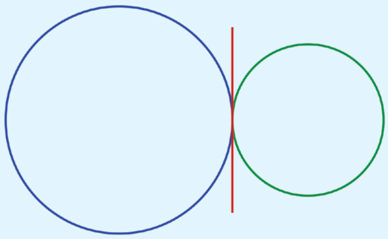
(iii) Prove that the perimeter of the triangle is equal to the diameter of the circle.

(2) The picture shows the triangle formed by three tangents to a circle:

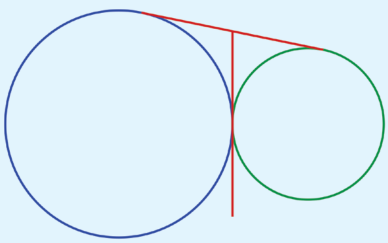


Calculate the length of each tangent from each vertex to the point of contact.

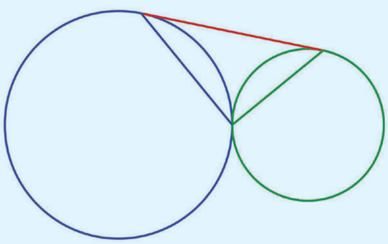
(3) The figure shows two circles touching at a point and the common tangent at this point:



(i) Prove that this tangent bisects another common tangent to these circles.



(ii) Show that the triangle formed by joining these points of contacts is right-angled.

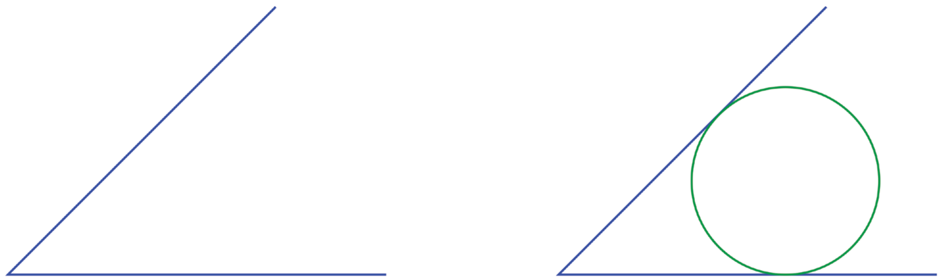


Circle touching a line

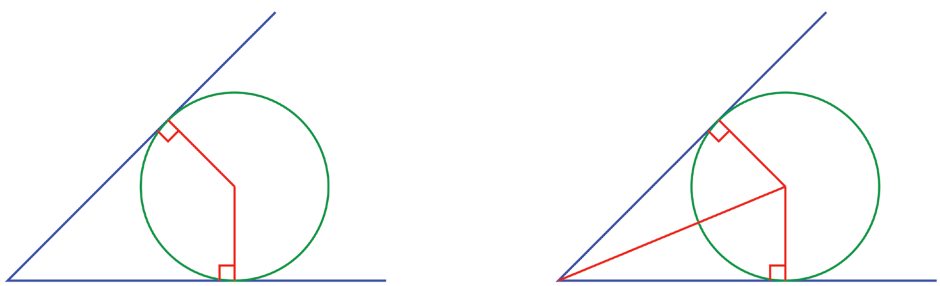
We have seen that two lines touching a circle can be drawn from a point outside the circle and also how we can draw them.

Now we ask the reverse question: Can we draw a circle touching two lines meeting at a point?

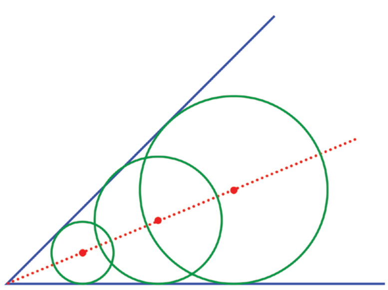
See this picture:



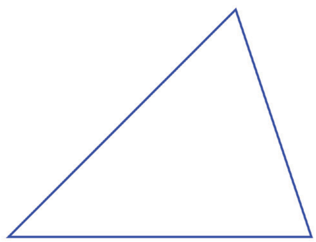
The perpendiculars from the centre of the circle to these lines are radii of the circle and hence equal. In other words, the centre of the circle is at equal perpendicular distances from these lines. So, the centre of the circle must be on the bisector of the angles between these lines (The section **Angle bisector** of the lesson **Bisectors** in the Class 8 textbook):

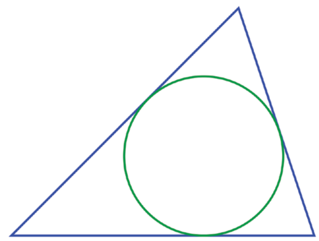


And we can draw a circle touching these lines with any point on the angle bisector as centre:



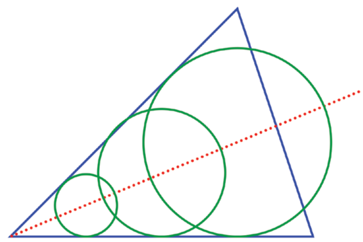
The next question is whether we can draw a circle touching all three sides of a triangle:



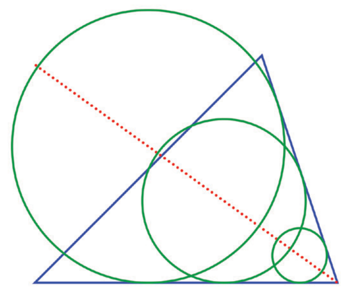


Draw an angle and its bisector in GeoGebra. Mark a point on the bisector and draw the perpendicular from it to a side of the angle. Mark the point of intersection. Draw a circle centred on the point on the bisector, passing through this point of intersection. Doesn't it touch the other side of the angle also? Change the position of the point on the bisector and check.

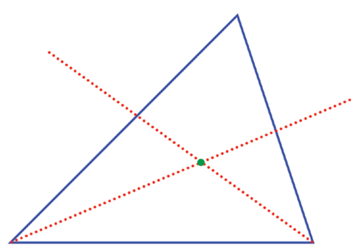
With any point on the bisector of the angle between the bottom and left sides as centre, we can draw a circle touching these sides:



If instead we take any point on the bisector of the angle between the bottom and right sides as centre, we can draw a circle touching those sides.

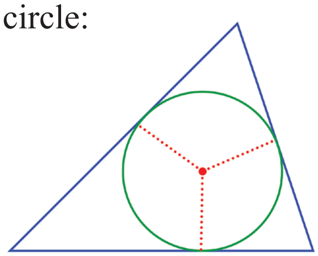


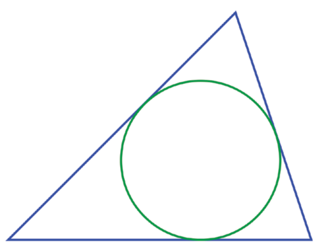
What if we take a point which is on both these bisectors, that is their point of intersection?



Draw a triangle in GeoGebra and draw its incircle.

The perpendiculars from this point to all three sides are equal, right? So, if we draw a circle with this point as centre and this length as radius, then all three sides would be tangents to the circle:





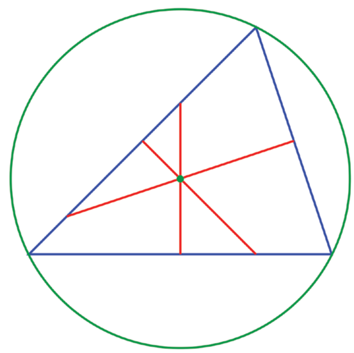
Such a circle within a triangle which touches all three sides is called the *incircle* of the triangle.

Draw a triangle in GeoGebra and draw its incircle.

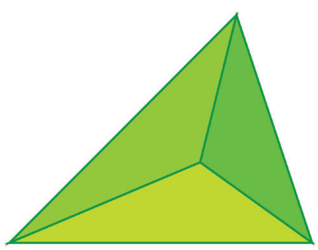
We can note another thing here. Since the incircle touches all three sides of the triangle, the perpendiculars from its centre to all three sides have equal length. This means, the centre of the incircle is on the bisectors of all three angles of the triangle. In other words, the bisectors of all three angles of the triangle pass through the centre of the incircle. This point is called the *incentre*.

In any triangle, the bisectors of all three angles intersect at a single point

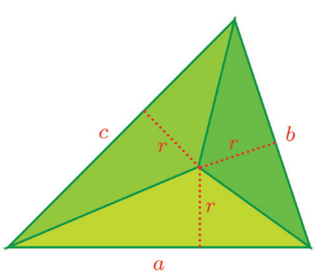
Recall what we saw in Class 8: in any triangle the perpendicular bisectors of all three sides intersect at a single point; and with this point as centre, a circle can be drawn passing through all three vertices of the triangle (the section **Circumcircle** of the lesson **Circles**):



There is a relation between the radius of the incircle and the area of the triangle. To see this, join the vertices of the triangle to the centre of the incircle, to split the triangle into three smaller triangles:

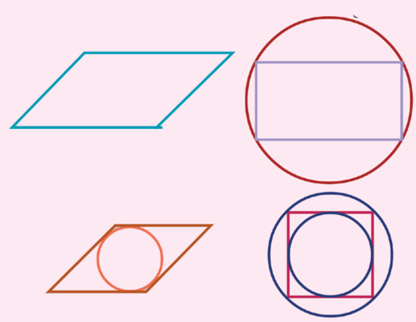


In each of these triangles, one side is a side of the original triangle; and the height from this is the radius of the incircle. So, if we take the lengths of the sides of the original triangle as a, b, c and the radius of its incircle as r, then the areas of the three smaller triangles are $\frac{1}{2}ar$, $\frac{1}{2}br$, $\frac{1}{2}cr$.



Circumcircle and incircle

We can draw circumcircle and incircle for any triangle. But when we come to quadrilaterals, some may have neither, some may have one and not the other, and some may have both:



The sum of these is equal to the area of the original triangle. Taking it as A, we have

$$A = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr = \frac{1}{2}(a+b+c)r$$

In this, $\frac{1}{2}(a+b+c)$ is half the perimeter of the triangle. We have seen in Class 9 that it is usually denoted as s (the section **Area of triangles** of the lesson **Irrational Multiplication**). So, the relation between the area of the triangle and the radius of its incircle can be written as

$$A = sr$$

which can also be written

$$r = \frac{A}{s}$$

The radius of the incircle of a triangle is equal to its area divided by half the perimeter

We have seen in Class 9 that the area of a triangle with lengths of sides a, b, c is

$$\sqrt{s(s-a)(s-b)(s-c)}$$

The radius of the incircle is this divided by s; that is,

$$r = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} = \frac{\sqrt{(s-a)(s-b)(s-c)}}{\sqrt{s}}$$

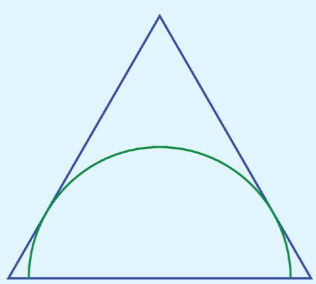
We can write this as

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$

(1) Draw the triangle with sides 7 centimetres, 8 centimetres, 9 centimetres and draw its incircle.

(2) Draw the rhombus with length of sides 5 centimetres and one angle 50° and draw its incircle.

(3) Draw an equilateral triangle and draw a semicircle touching two of its sides as shown in the figure:



(4) Calculate the radius of the incircle of the equilateral triangle with side 12 centimetres.

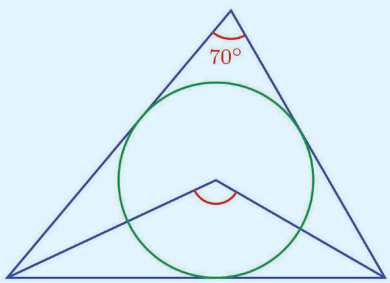
(5) Calculate the radius of the incircle of a triangle with sides 13, 14 and 15 centimetres.

(6) Prove that the radius of the incircle of any equilateral triangle is half the radius of its circumcircle.

(7) Prove that if the hypotenuse of a right triangle is h and the radius of its incircle is r, then its area is $r(h + r)$.

(8) The lengths of the perpendicular sides of a right triangle are a, b and its hypotenuse is c. Prove that the radius of the incircle is $\frac{1}{2}(a + b - c)$.

(9) In the picture, two vertices of a triangle are joined to its incentre:



(i) Calculate the angle marked in the picture.

(ii) In any triangle, what is the relation between the angle at a vertex, and the angle between the lines joining the other two vertices to the incentre?

(iii) If one of these angles is half the other, what is the angle of the triangle?

9 POLYNOMIALS AND EQUATIONS

Multiplications

We have seen different easy ways to calculate the product of two numbers. Can you do this multiplication in head?

$$38 \times 32$$

How about writing 38 as 30 + 8 and 32 as 30 + 2?

Remember what we said about the product of two sums in the lesson **Multiplication Identities** in Class 9?

> To multiply a sum by a sum, each number in one sum is to be multiplied by each number in the other sum, and the products added.

We also wrote this in shorthand as an algebraic equation:

> For any four numbers x, y, u, v

$$(x + y)(u + v) = xu + xv + yu + yv$$

Using this,

$$\begin{aligned} 38 \times 32 &= (30 + 8) \times (30 + 2) \\ &= (30 \times 30) + (30 \times 2) + (8 \times 30) + (8 \times 2) \end{aligned}$$

In this, the sum $(30 \times 2) + (8 \times 30)$ of the middle two products can be combined like this:

$$\begin{aligned} (30 \times 2) + (8 \times 30) &= (2 \times 30) + (8 \times 30) \\ &= (2 + 8) \times 30 \\ &= 10 \times 30 \\ &= 300 \end{aligned}$$

Thus we get

$$38 \times 32 = (30 \times 30) + (30 \times 2) + (8 \times 30) + (8 \times 2)$$
$$= 900 + 300 + 16$$
$$= 1216$$

Can't we do 48×42 like this?

$$48 \times 42 = 40^2 + ((8 + 2) \times 40) + (8 \times 2)$$
$$= 1600 + 400 + 16$$
$$= 2016$$

How about 78×72?

$$78 \times 72 = 4900 + 700 + 16 = 5616$$

We can write the multiplications done in all three examples above as a general principle:

For any number x

$$(x + 2)(x + 8) = x^2 + (8 + 2)x + 16$$
$$= x^2 + 10x + 16$$

What if we take some other numbers instead of 2 and 8 in this?

Do the multiplications below in head. Write the general principle got from each set as an algebraic identity:

(1) (i) 43×47 (ii) 63×67 (iii) 103×107

(2) (i) 51×52 (ii) 81×82 (iii) 301×302

Polynomial multiplication

What general principle do we get from the second set of problems above?

For any natural number x

$$(x + 1)(x + 2) = x^2 + 3x + 2$$

According to what we noted in the beginning about the product of sums, this is true not only for a natural number x, but for any number whatsoever. That is,

$$(x + 1)(x + 2) = x^2 + 3x + 2 \text{ for any number } x$$

In this, when we take different numbers as x, we get many numbers as $x + 1$, $x + 2$ and $x^2 + 3x + 2$; and this equation shows the relation between these numbers.

We can consider all these algebraic expressions as polynomials. Then the above equation can be interpreted like this:

The product of the first degree polynomials $x + 1$ and $x + 2$ is the second degree polynomial $x^2 + 3x + 2$.

Like this, can you state the algebraic identity got from the first set of problems in terms of polynomial multiplication?

Let's look at the algebraic identities got from the different problems done so far:

- $(x + 2)(x + 8) = x^2 + 10x + 16$
- $(x + 3)(x + 7) = x^2 + 10x + 21$
- $(x + 1)(x + 2) = x^2 + 3x + 2$

In all these, what are the relations between the numbers in the first degree polynomials multiplied and the second degree polynomial got as the product?

Can you write a very general algebraic identity which includes all these products?

For any numbers x, a, b

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

How do we state this in terms of polynomials?

The product of the first degree polynomials $x + a$ and $x + b$ is the second degree polynomial $x^2 + (a + b)x + ab$

How do we write $(x + 5)(x + 3)$ using this?

$$(x + 5)(x + 3) = x^2 + 8x + 15$$

We have seen general identities on not only the product of sums, but also on the product of a sum and difference and product of differences in the lesson **Multiplication Identities** in the Class 9 textbook:

- $(x + y)(u - v) = xu - xv + yu - yv$
- $(x - y)(u - v) = xu - xv - yu + yv$

Using these, we can write the products of some other kinds of first degree polynomials:

- $(x + a)(x - b) = x^2 + (a - b)x - ab$
- $(x - a)(x - b) = x^2 - (a + b)x + ab$

For example,

$$(x + 4)(x - 1) = x^2 + 3x - 4$$

We can do this in another way. We can write

$$x - 1 = x + (-1)$$

so that

$$(x + 4)(x - 1) = (x + 4)(x + (-1))$$

Now in the identity

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

for the multiplication of sums, if we take a as 4 and b as -1, we get

$$(x + 4)(x + (-1)) = x^2 + (4 + (-1))x + (4 \times (-1))$$

$$= x^2 + 3x - 4$$

Thus we get as before

$$(x + 4)(x - 1) = x^2 + 3x - 4$$

Like this, the product $(x - 4)(x - 1)$ can be computed in two ways

In the identity

$$(x - a)(x - b) = x^2 - (a + b)x + ab$$

for the multiplication of differences, take a as 4 and b as 1 to get

$$(x - 4)(x - 1) = x^2 - 5x + 4$$

When we write the difference $x - 4$ as the sum $x + (-4)$ and difference $x - 1$ as the sum $x + (-1)$ we get,

$$(x - 4)(x - 1) = (x + (-4))(x + (-1))$$

Or in the identity

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

for the multiplication of sums, take a as -4 and b as -1 to get

$$(x - 4)(x - 1) = (x + (-4))(x + (-1))$$

$$= x^2 + ((-4) + (-1))x + ((-4) \times (-1))$$

$$= x^2 - 5x + 4$$

Find the following products

(i) $(x + 2)(x + 5)$ (ii) $(x + 2)(x - 5)$ (iii) $(x - 2)(x + 5)$ (iv) $(x - 2)(x - 5)$

Polynomial factors

We have done several problems on calculating the product of two first degree polynomials as a second degree polynomial. Now a question in reverse:

Can we write $x^2 + 5x + 6$ as the product of two first degree polynomials?

Here the product is given and we have to find the polynomials to be multiplied; thus we can rephrase the question like this:

To get $(x + a)(x + b) = x^2 + 5x + 6$, what numbers should we take as a and b?

We can write the product $(x + a)(x + b)$ as

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

So, the question can be written like this:

To get $x^2 + (a + b)x + ab = x^2 + 5x + 6$ what numbers should we take as a and b?

For this equation to hold, it is enough if we have $a + b = 5$ and $ab = 6$. So, we can change the question to this:

To get $a + b = 5$ and $ab = 6$, what numbers should we take as a and b?

In ordinary language, this means,

What two numbers have sum 5 and product 6?

A little thought would give the numbers as 2 and 3. So, the answer to the very first question is

$$x^2 + 5x + 6 = (x + 2)(x + 3)$$

Suppose we slightly change our original question like this:

Write $x^2 - 5x + 6$ as the product of two first degree polynomials.

As in the first problem, we can rewrite the question step by step:

(i) To get $(x + a)(x + b) = x^2 - 5x + 6$, what numbers should we take as a, b?

(ii) To get $x^2 + (a + b)x + ab = x^2 - 5x + 6$, what numbers should we take as a and b?

What can we say about $a + b$ and ab from this?

So, we can continue our thoughts like this:

(iii) To get $a + b = -5$ and $ab = 6$ what numbers should we take as a and b?

(iv) What two numbers have sum -5 and product 6?

If we take as before the numbers 2 and 3 giving the product 6, the sum would not be -5

How do we write 6 as a product of negative numbers?

$$6 = (-2) \times (-3)$$

(The section **Time and distance** of the lesson, **Negative Numbers** in the Class 9 textbook).

Moreover, we do have

$$(-2) + (-3) = -2 - 3 = -5$$

(The section **Position and number** of the lesson, **Negative Numbers** in the Class 9 textbook).

Thus the numbers in our last question above are -2 and -3.

And the answer to the first question is

$$\begin{aligned} x^2 - 5x + 6 &= (x + (-2))(x + (-3)) \\ &= (x - 2)(x - 3) \end{aligned}$$

Now what if we change the polynomial in the question like this?

Write $x^2 + 5x - 6$ as the product of two first degree polynomials.

How do we rewrite the question step by step?

(i) To get $(x + a)(x + b) = x^2 + 5x - 6$, what numbers should we take as a and b?

(ii) To get $x^2 + (a + b)x + ab = x^2 + 5x - 6$, what numbers should we take as a and b?

(iii) To get $a + b = 5$ and $ab = -6$, what numbers should we take as a and b?

(iv) What two numbers have sum 5 and product -6?

Now if we write $-6 = (-2) \times 3$ or $-6 = 2 \times (-3)$, we don't get the sum of the factors as -5.

So we must write 6 as the product of two other factors.

Using the fact that $6 = 1 \times 6$, we get these factorizations:

$$-6 = (-1) \times 6$$
$$-6 = 1 \times (-6)$$

If we take the first of these, we do get the sum of the factors as 5.

Thus the numbers asked for in question (iv) above are 6 and −1 and the answer to the question about polynomial factorization is

$$x^2 + 5x - 6 = (x + 6)(x - 1)$$

Now can't you find the answer to this question without much thought?

Write $x^2 - 5x - 6$ as the product of two first degree polynomials

$$x^2 - 5x - 6 = (x - 6)(x + 1)$$

Like this, write the second degree polynomials below as the product of two first degree polynomials:

Unification

To calculate the first degree factors of both $x^2 + 5x + 6$ and $x^2 - 5x + 6$, we wrote the polynomial in the form

$$(x + a)(x + b)$$

and proceeded to compute a and b

In the first problem, we got $a = 2$ and $b = 3$, both positive numbers.

In the second problem, we got $a = -2$ and $b = -3$, both negative.

In the third problem, we got $a = 6$ and $b = -1$, a positive, b negative.

In the fourth problem, we got $a = -6$ and $b = 1$, a negative, b positive.

In short, to find the first degree factors of any second degree polynomial, we need only the single identity

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

We have this convenience because we use negative numbers also.

(i) $x^2 + x - 6$ (ii) $x^2 - x - 6$ (iii) $x^2 + 7x + 6$ (iv) $x^2 - 7x + 6$

In all these, we found the numbers in the first degree polynomials simply by mental computations.

This may not be possible always. For example, see this problem:

Write $x^2 + 27x + 180$ as the product of two first degree polynomials.

To do this, we first write

$$(x + a)(x + b) = x^2 + 27x + 180$$

That is,

$$x^2 + (a + b)x + ab = x^2 + 27x + 180$$

From this we get

$$a + b = 27$$
$$ab = 180$$

If we go by the method used in the problems so far, we have to factorize 180 as the product of two numbers in different ways, and check which pair gives the sum 27; but this is not very easy. So, let's see if there is a different method.

Here we know the sum and product of two numbers. If it were the sum and difference, we could have found the numbers (the last part of the section **Two equations** of the lesson **Pairs of Equations** in the Class 9 textbook).

Can we calculate the difference of two numbers using their sum and product?

Haven't we seen in Class 8, how we can calculate the square of the difference of two numbers from the square of their sum and product? (The section **Square of difference** in the lesson **Square Identities**).

$$(a-b)^2 = (a+b)^2 - 4ab$$

Using this in our current problem,

$$\begin{aligned}(a-b)^2 &= 27^2 - (4 \times 180)\\ &= (400 + 280 + 49) - (72 \times 10)\\ &= 729 - 720\\ &= 9\end{aligned}$$

So what numbers can $a - b$ be?

$$a - b = 3 \text{ or } a - b = -3$$

Taking $a - b = 3$, we have

$$\begin{aligned}a + b &= 27\\ a - b &= 3\end{aligned}$$

From these we get

$$\begin{aligned}a &= \tfrac{1}{2}(27 + 3) = \tfrac{1}{2} \times 30 = 15\\ b &= \tfrac{1}{2}(27 - 3) = \tfrac{1}{2} \times 24 = 12\end{aligned}$$

If we take $a - b = -3$, we get instead

$$\begin{aligned}a + b &= 27\\ a - b &= -3\end{aligned}$$

and from these we get

$$\begin{aligned}a &= \tfrac{1}{2}(27 - 3) = \tfrac{1}{2} \times 24 = 12\\ b &= \tfrac{1}{2}(27 + 3) = \tfrac{1}{2} \times 30 = 15\end{aligned}$$

In either case, we get

$$x^2 + 27x + 180 = (x + 12)(x + 15)$$

In general, if we can write the polynomial $p(x)$ as $p(x) = q(x)r(x)$, the product of two polynomials $q(x)$ and $r(x)$, then the polynomials $q(x)$ and $r(x)$ are called *factors* of the polynomial $p(x)$ (In the case of numbers also, the meaning of factors is the same, isn't it? For example, since $10 = 2 \times 5$, we call 2 and 5 as factors of 10).

So, in all the problems above, what we have done is to factorize a second degree polynomial as the product of two first degree factors.

Write the polynomials below as the product of two first degree factors:

(i) $x^2 + 30x + 221$ (ii) $x^2 + 4x - 221$ (iii) $x^2 + x - 156$

Factors and solutions

Let's look at a problem we did in the section **Two answers** of the lesson **Second degree equations**

To get $x^2 - 4x + 3 = 0$, what number should we take as x?

How did we do this?

(i) $x^2 - 4x + 3 = 0$

(ii) $x^2 - 4x + 4 = 1$

(iii) $(x - 2)^2 = 1$

(iv) $x - 2 = 1$ or $x - 2 = -1$

(v) $x = 3$ or $x = 1$

We can do this in another way. For that, we first write the polynomial $x^2 - 4x + 3$ in the problem as the product of two first degree polynomials.

This we can do in head, right?

$$x^2 - 4x + 3 = (x - 1)(x - 3)$$

So we can rewrite our problem like this:

To get $(x-1)(x-3)=0$, what number should we take as x ?

For the product of two numbers to be zero, at least one of them should be zero, right?

So, to get $(x-1)(x-3)=0$, we should have either $x-1=0$ or $x-3=0$

In other words, either we must take $x=1$ or $x=3$

We can condense the steps of this procedure like this:

(i) $x^2-4x+3=0$

(ii) $x^2-4x+3=(x-1)(x-3)$

(iii) $(x-1)(x-3)=0$

(iv) $x-1=0$ or $x-3=0$

(v) $x=1$ or $x=3$

Let's slightly change the problem like this:

To get $x^2-5x+6=0$, what number should we take as x?

How do we write x^2-5x+6 as the product of two first degree polynomials?

$$x^2-5x+6=(x-2)(x-3)$$

Now how do we write the steps leading to the solution?

(i) $x^2-5x+6=0$

(ii) $x^2-5x+6=(x-2)(x-3)$

(iii) $(x-2)(x-3)=0$

(iv) $x-2=0$ or $x-3=0$

(v) $x=2$ or $x=3$

In the first problem, we found the solutions of the second degree equation $x^2-4x+3=0$ as 1 and 3; in the second problem, we found the solutions of the second degree equation $x^2-5x+6=0$ as 2 and 3

Thus we can solve any second degree equation by factorizing the polynomial, instead of completing the square as done earlier.

For example, see this problem:

> One side of a rectangle is 3metres longer than the other side and its area is 270 square metres. What are the lengths of the sides?

If we denote the length of the shorter side as x metres, then the length of the longer side is $x + 3$ metres and its area is

$$x(x + 3) = x^2 + 3x \text{ square metres.}$$

So the problem can be stated using algebra like this:

To get $x^2 + 3x = 270$, what number should we take as x?

To do this using the earlier method of completing the square, we must add $\left(\frac{3}{2}\right)^2$ and proceed.

Let's try the factorization method instead.

To do this, we first rewrite the problem like this:

To get $x^2 + 3x - 270 = 0$, what number should we take as x?

Next we must factorize $x^2 + 3x - 270$. For this we write

$$x^2 + 3x - 270 = (x + a)(x + b) = x^2 + (a + b)x + ab$$

and from this get

$$a + b = 3$$
$$ab = -270$$

We now compute $a - b$ using these:

$$(a - b)^2 = (a + b)^2 - 4ab = 9 - (4 \times (-270)) = 9 - (-1080) = 9 + 1080 = 1089$$

To get $a - b$ we must calculate the square root of 1089. We have seen how square roots can be computed through factorization in class 7 (the section **Areas of squares** of the lesson **Squares And Right Triangles**).

Since the sum of the digits of 1089 is 18 which is a multiple of 9, the number itself is a multiple of 9 (the section **Digit sum** of the lesson **Number Relation** in the Class 6 textbook).

So, let's compute

$$1089 \div 9$$

We have done such computations in Class 5.

	9	
	1089−	
100	900	$9 \times 100 = 900$
	189−	
20	180	$9 \times 20 = 180$
	9−	
1	9	$9 \times 1 = 9$
$1089 \div 9 = 121$	0	$9 \times 121 = 1089$

Thus we get

$$1089 = 9 \times 121$$

This we can write like this:

$$1089 = 3^2 \times 11^2 = (3 \times 11)^2 = 33^2$$

(The section **Multiples and Powers** of the chapter **Repeated Multiplication** in the Class 7 textbook). What can we say from the equation $(a - b)^2 = 33^2$?

$$a - b = 33 \text{ or } a - b = -33$$

If we take $a - b = 33$, we get

$$a + b = 3$$

$$a - b = 33$$

From this we get

$$a = \frac{1}{2}(3 + 33) = \frac{1}{2} \times 36 = 18$$

$$b = \frac{1}{2}(3 - 33) = \frac{1}{2} \times (-30) = -15$$

Thus we have

$$x^2 + 3x - 270 = (x - 15)(x + 18)$$

So our problem can be now rewritten like this:

To get $(x - 15)(x + 18) = 0$, what number should we take as x?

As seen in earlier problems, the answer to this is $x = 15$ or $x = -18$

In our problem, x is the length of a side of a rectangle and so cannot be a negative number. So, the length of the shorter side of the rectangle is 15 metres.

The length of the longer side is $15 + 3 = 18$ metres.

Check what we get if in this problem if we proceed with $a - b = -33$ instead of $a - b = 33$.

Let's look at another problem:

One added to a number gives the square of the number. What is the number?

Taking the number as x, the problem can be written in algebra like this:

To get $x + 1 = x^2$, what number should we take as x?

To get $x + 1 = x^2$, we must have $x^2 - (x + 1) = 0$. In this,

$$x^2 - (x + 1) = x^2 - x - 1$$

(We have seen in class 7 that instead of subtracting a sum, we can subtract the addends one by one, in the section **One by one and altogether** of the lesson **Shorthand Math**).

So, we can rewrite our question like this:

To get $x^2 - x - 1 = 0$, what number should we take as x?

Next we factorize $x^2 - x - 1$ by writing

$$x^2 - x - 1 = (x + a)(x + b) = x^2 + (a + b)x + ab$$

and from this we get

$$a + b = -1$$
$$ab = -1$$

So we can calculate

$$(a - b)^2 = (a + b)^2 - 4ab = (-1)^2 - (4 \times (-1))$$
$$= 1 - (-4) = 1 + 4 = 5$$

and this gives

$$a - b = \sqrt{5} \text{ or } a - b = -\sqrt{5}$$

Taking $a - b = \sqrt{5}$ we get

$$a + b = -1$$
$$a - b = \sqrt{5}$$

Now we can compute a and b as

$$a = \frac{1}{2}\left(-1 + \sqrt{5}\right) = \frac{\sqrt{5} - 1}{2}$$
$$b = \frac{1}{2}\left(-1 - \sqrt{5}\right) = -\frac{\sqrt{5} + 1}{2}$$

Thus we get

$$x^2 - x - 1 = \left(x + \frac{\sqrt{5} - 1}{2}\right)\left(x - \frac{\sqrt{5} + 1}{2}\right)$$

Role reversal

We write a second degree polynomial $p(x)$ as the product of two first degree polynomials

$$p(x) = (x + a)(x + b)$$

and compute a and b.

In such problems, we first get $a + b$ and then $(a - b)^2$. If they are of the form

$$a + b = h$$
$$(a - b)^2 = k^2$$

We can take $a - b$ as either k or $-k$. If we take $a + b = h$ and $a - b = k$, we get

$$a = \frac{1}{2}(h + k)$$
$$b = \frac{1}{2}(h - k)$$

What if we take $a - b = -k$?

$$a = \frac{1}{2}(h - k)$$
$$b = \frac{1}{2}(h + k)$$

Thus the numbers a and b are interchanged; the factors however do not change.

Now the question becomes this:

$$\text{To get } \left(x+\frac{\sqrt{5}-1}{2}\right)\left(x-\frac{\sqrt{5}+1}{2}\right)=0 \text{ what should we take as } x?$$

The answer to this is

$$x=-\frac{\sqrt{5}-1}{2} \quad \text{or} \quad x=\frac{\sqrt{5}+1}{2}$$

This means the speciality of the positive number $\frac{\sqrt{5}+1}{2}$ and the negative number $-\frac{\sqrt{5}-1}{2}$ is that 1 added to either of them gives its square.

(1) One side of a rectangle is 2 metres longer than the other and its area is 48 square metres. What are the lengths of its sides?

(2) One side of a right triangle is one centimetre less than twice the length of the side perpendicular to it; the hypotenuse is one centimetre more than twice this side. What are the lengths of the sides?

(3) The sum of the consecutive natural numbers from 1up to which number is 300?

(4) At what points does the graph of the polynomial $x^2 - 2x - 1$ cross the x-axis?

General solution

See this problem:

At what points does the graph of the polynomial $2x^2 + 3x - 2$ cross the x-axis?

We can rephrase the question like this:

To get $2x^2 + 3x - 2 = 0$, what number should we take as x?

This is a bit different from similar problems we have done earlier. In all the problems done so far we had only x^2 in the polynomial; here it is $2x^2$

So we first write $2x^2 + 3x - 2$ as

$$2x^2 + 3x - 2 = 2\left(x^2+\frac{3}{2}x-1\right)$$

From this we can see that if we take any number x which makes $x^2+\frac{3}{2}x-1=0$; also makes twice this polynomial $2x^2 + 3x - 2 = 0$; on the other hand, any number x which makes $2x^2 + 3x - 2 = 0$, also makes half this polynomial $x^2+\frac{3}{2}x-1=0$.

So we can again rewrite our problem like this:

To get $x^2 + \frac{3}{2}x - 1 = 0$, what number should we take as x?

Now we can find x as before using factorization. Writing

$$x^2 + \frac{3}{2}x - 1 = (x + a)(x + b) = x^2 + (a + b)x + ab$$

we get

$$a + b = \frac{3}{2}$$

$$ab = -1$$

and from this

$$(a - b)^2 = (a + b)^2 - 4ab = \frac{9}{4} - (4 \times (-1)) = \frac{9}{4} + 4 = \frac{25}{4}$$

So we have

$$a - b = \frac{5}{2} \text{ or } a - b = -\frac{5}{2}$$

Taking $a - b = \frac{5}{2}$, we get

$$a + b = \frac{3}{2}$$

$$a - b = \frac{5}{2}$$

and this gives

$$a = \frac{1}{2} \times \left(\frac{3}{2} + \frac{5}{2}\right) = \frac{1}{2} \times 4 = 2$$

$$b = \frac{1}{2} \times \left(\frac{3}{2} - \frac{5}{2}\right) = \frac{1}{2} \times (-1) = -\frac{1}{2}$$

Thus we find

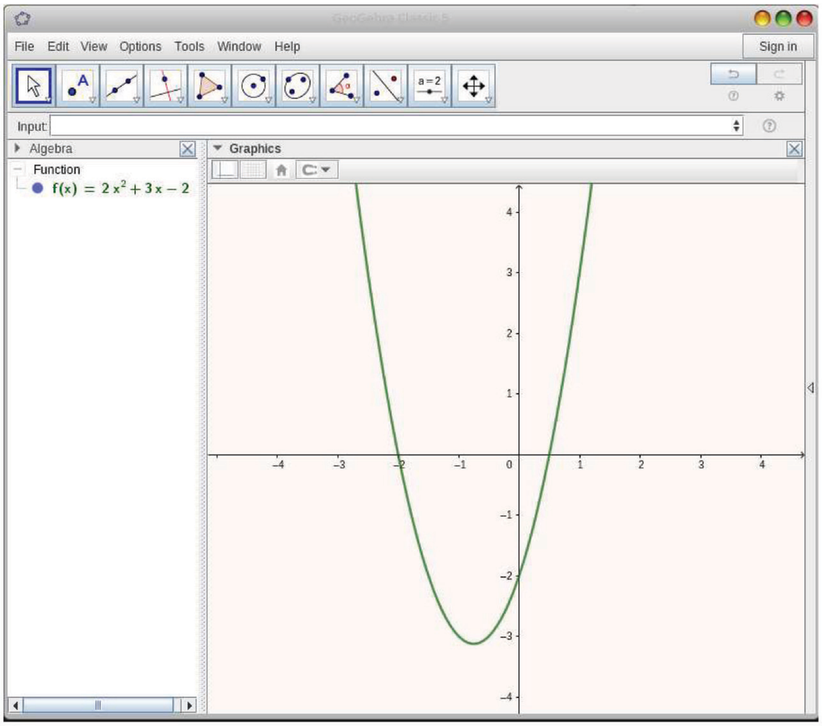
$$x^2 + \frac{3}{2}x - 1 = (x + 2)\left(x - \frac{1}{2}\right)$$

So, the problem becomes this:

To get $(x + 2)\left(x - \frac{1}{2}\right) = 0$, what number should we take as x?

The answer to this is $x = -2$ or $x = \frac{1}{2}$.

Thus the points where the graph of the polynomial $2x^2 + 3x - 2$ crosses the x-axis are $(-2, 0)$ and $\left(\frac{1}{2}, 0\right)$.



Now let's look at this method in general terms:

Any second degree polynomial can be written in the form $ax^2 + bx + c$. So any second degree equation can be written in the form

$$ax^2 + bx + c = 0$$

The method of computing the solutions by factorization can be split into stages like this:

(1) Change the equation to $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

(2) Write the polynomial in this as the product of first degree factors:

(i) Writing the factors as $x + p$ and $x + q$, we write

$$x^2 + \frac{b}{a}x + \frac{c}{a} = (x + p)(x + q) = x^2 + (p + q)x + pq$$

(ii) From this we get $p + q = \frac{b}{a}$ and $pq = \frac{c}{a}$.

(iii) Use the identity $(p - q)^2 = (p + q)^2 - 4pq$ to compute $p - q$.

(iv) Use $p + q$ found earlier and one of the values of $p - q$ to calculate p and q.

(3) Then the equation to be solved can be written as $(x + p)(x + q) = 0$.

(4) The solutions to this are $x = -p$ or $x = -q$.

Now let's see in detail how $p - q$ and then p and q are calculated:

$$(p-q)^2 = \left(\frac{b}{a}\right)^2 - 4\left(\frac{c}{a}\right)$$
$$= \frac{b^2}{a^2} - \frac{4c}{a} = \frac{b^2}{a^2} - \frac{4ac}{a^2} = \frac{b^2 - 4ac}{a^2}$$
$$p - q = \frac{\sqrt{b^2 - 4ac}}{a} \text{ or } p - q = -\frac{\sqrt{b^2 - 4ac}}{a}$$

We can calculate p and q by taking either of these as $p - q$.

$$p + q = \frac{b}{a} \qquad p - q = \frac{\sqrt{b^2 - 4ac}}{a}$$
$$p = \frac{1}{2}\left(\frac{b}{a} + \frac{\sqrt{b^2 - 4ac}}{a}\right) = \frac{1}{2} \times \frac{b + \sqrt{b^2 - 4ac}}{a} = \frac{b + \sqrt{b^2 - 4ac}}{2a}$$
$$q = \frac{1}{2}\left(\frac{b}{a} - \frac{\sqrt{b^2 - 4ac}}{a}\right) = \frac{1}{2} \times \frac{b - \sqrt{b^2 - 4ac}}{a} = \frac{b - \sqrt{b^2 - 4ac}}{2a}$$

Thus we get the solutions of the equation as

$$x = -\frac{b + \sqrt{b^2 - 4ac}}{2a} \text{ or } x = -\frac{b - \sqrt{b^2 - 4ac}}{2a}$$

This we can write as

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \text{ or } x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

We use the symbol ± to denote adding or subtracting a number. So, we can write the solutions like this:

The solutions of the equation

$$ax^2 + bx + c = 0$$

are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

As an illustration let's have another look at an earlier problem:

At what points does the graph of the polynomial $2x^2 + 3x - 2$ cross the x-axis?

Here the equation to be solved is

$$2x^2 + 3x - 2 = 0$$

So we take in the general form of the solutions stated above

$$a = 2 \qquad b = 3 \qquad c = -2$$

to get the solutions as

$$x = \frac{-3 \pm \sqrt{3^2 - (4 \times 2 \times (-2))}}{2 \times 2}$$
$$= \frac{-3 \pm \sqrt{9+16}}{4}$$
$$= \frac{-3 \pm \sqrt{25}}{4}$$
$$= \frac{-3 \pm 5}{4}$$

That is either

$$x = \frac{-3+5}{4} = \frac{2}{4} = \frac{1}{2}$$

or

$$x = \frac{-3-5}{4} = \frac{-8}{4} = -2$$

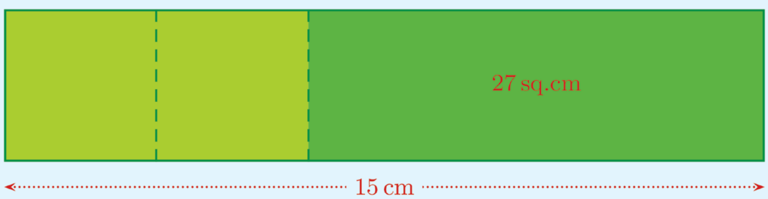
So the points where the graph crosses x - axis are $\left(\frac{1}{2}, 0\right), (-2, 0)$.

(1) Find the points at which the graphs of the polynomials given below cross the x-axis:

(i) $2x^2 - 7x - 1$ (ii) $2x^2 + 7x - 1$ (iii) $9x^2 + 12x + 4$

(2) The perimeter of a rectangle is 42 metres and its diagonal is 15 metres. What are the lengths of its sides?

(3) From a rectangular sheet of paper, two squares are cut off as shown below:



The area of remaining part is 27 square centimetres. What is the length of the shorter side of the rectangle?

(4) How many terms of the arithmetic sequence, 1, 5, 9, ... starting from the first, are to be added to get 91?

(5) A rectangle is to be made by bending a 28 centimetres long rod

(i) Can a rectangle of diagonal 8 centimetres be made?

(ii) How about a rectangle of diagonal 10 centimetres?

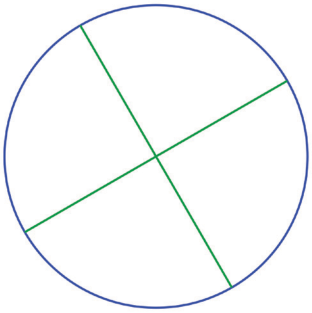
(iii) And a rectangle of diagonal 14 centimetres?

Calculate the lengths of the sides of those rectangles which can be made as above.

10 CIRCLES AND LINES

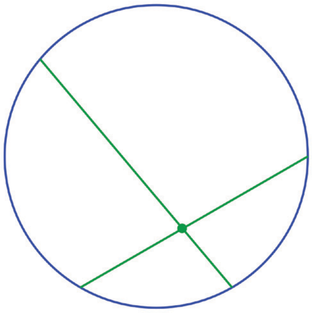
Chords

Any two diameters of a circle intersect at the centre of the circle:

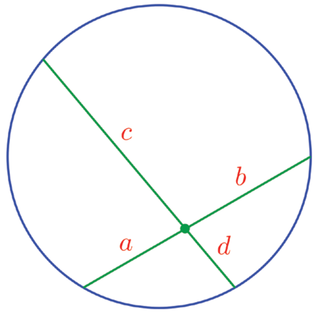


All four parts made by the intersection are equal to the radius of the circle.

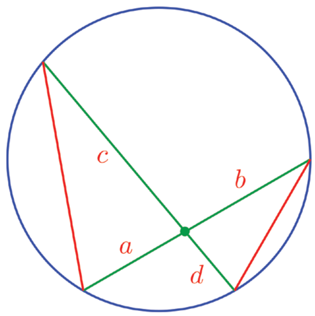
What happens when two chords, which are not diameters, intersect?



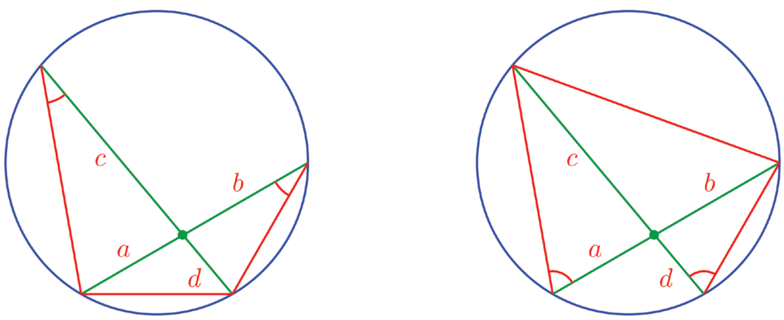
The parts may not be equal. But there is a relation between them. To see it, let's denote the lengths of the parts of one chord as *a, b* and the lengths of the parts of the other as *c, d*:



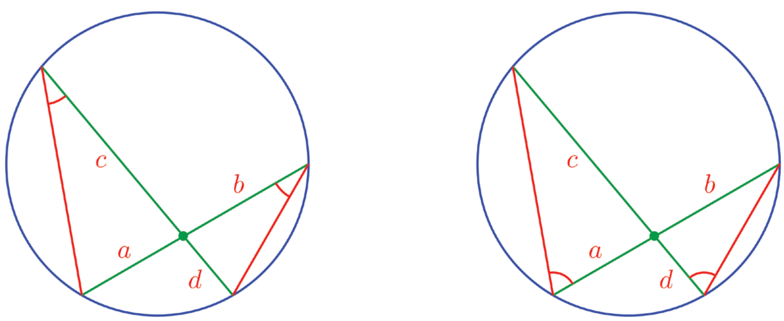
To find the relation between the parts, we join the ends of the chords to make two triangles as shown below:



The top angles of the triangles are angles in a segment of the circle; and the bottom angles are angles in another segment:



So the top angles of the triangles in the first picture are equal; and so are the bottom angles:



This means the triangles have the same angles. So, the lengths of the sides opposite the equal angles are scaled by the same factor (the section **Angles and sides** of the lesson **Similar Triangles** in the Class 9 textbook).

The lengths of the sides opposite the top angles are a and d; and the lengths of the sides opposite the bottom angles are c and b.

They are scaled by the same factor. If we denote the scale factor as k, then we get

$$d = ka$$

$$b = kc$$

We don't know the value of k. To eliminate k and get a relation between $a, b, c, d,$ we first calculate from the first equation

$$k = \frac{d}{a}$$

Using this in the second equation, we get

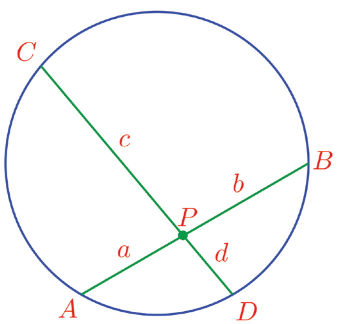
$$b = \frac{d}{a} \times c = \frac{cd}{a}$$

We can rewrite this as

$$ab = cd$$

If we name the chords AB and CD and their point of intersection P, we can write this as

$$AP \times PB = CP \times PD$$



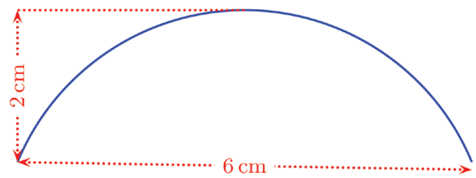
Draw a circle and two chords AB and CD in GeoGebra. Mark the point P where the chords intersect. If we type PA * PB as Input, we get the value of PA × PB in the Algebra view. Compute PC × PD also like this. Compare these products. What happens to these when P is on the circle?

We can state this as a general result:

When two chords of a circle intersect within the circle, the product of the parts of one chord is equal to the product of the parts of the other

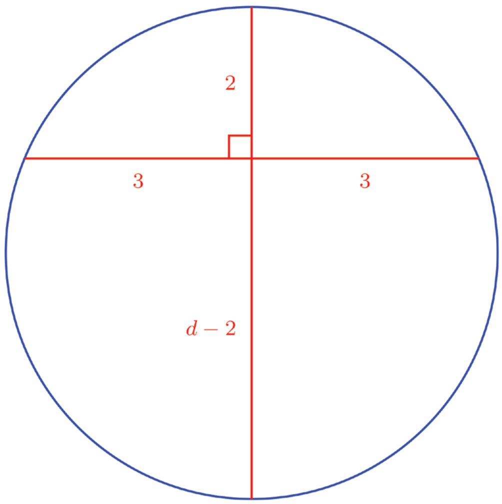
Let's look at a problem using this:

The picture shows a part of a ring:



Calculate the diameter of the ring.

Let's imagine the full circle, the chord joining the ends of this arc and the diameter perpendicular to it. This diameter bisects the chord, since it is perpendicular to the chord. So taking the diameter as d centimetres, we get the picture below:



Now using the principle stated above, we get

$$2(d-2) = 3 \times 3 = 9$$

From this we get

$$d = \frac{9}{2} + 2 = 4\frac{1}{2} + 2 = 6\frac{1}{2}$$

Thus the diameter of the ring is $6\frac{1}{2}$ centimetres.

Lotus problem

We have mentioned the book *lilavati* of Bhaskaraacharya in the Class 8 textbook (the section *Inverse operations* of the lesson *Solutions of Equations*. The translation of a *sloka* in it is this:

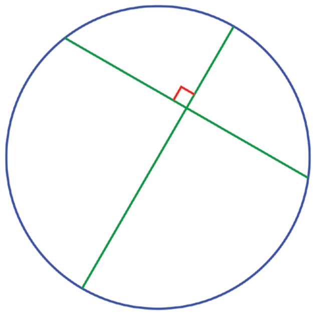
> In a lake full of frolicking chakravaka and krauncha birds, a lotus bud stands erect, half a hand-span high. Swaying gently in the breeze, it submerged, two hand-span away. Oh reckoner!, quickly tell the depth of the lake.

Try to find the answer of this question, as you found the diametre of the ring:

The actual sloka is:

chakrakraunchaakulitasalilee
kvaapidrshtam tadaagee
toyaaduurdhva kamalakalikagram
vitasti pramaanam
mandam mandam chalitamanileenaahatam
hastayugme
tasminmagnam ganaka, kathaya
kshipramambupramaanam

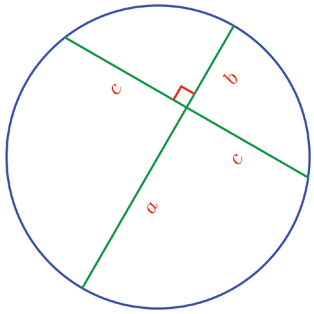
In this problem, one of the intersecting chords is a diameter of the circle and the other is a chord perpendicular to it. See such a picture:



Let's denote the lengths of the parts of the diameter as *a* and *b*.

Since the diameter is the perpendicular from the centre of the circle to a chord, it bisects the chord (the section **Chords** of the lesson **Circles** in the Class 8 textbook).

So, we can denote the lengths of both parts of the chord as *c*:



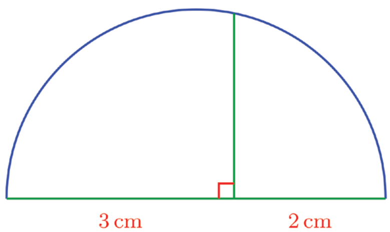
What is the relation between *a, b* and *c* according to our general principle?

$$ab = c^2$$

This we can state like this:

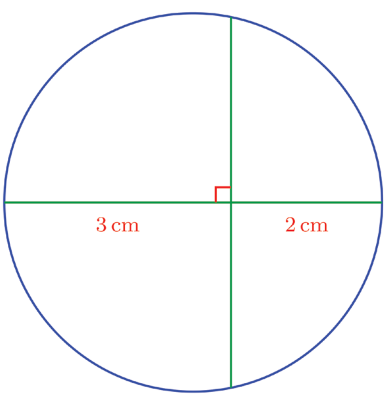
> The product of the parts into which a diameter of a circle is cut by a perpendicular chord, is the square of half the chord

For example, see this picture:



The perpendicular from a point on the diameter meets the semicircle. What is the length of this perpendicular?

If we draw the complete circle and extend the perpendicular downwards, it becomes a chord and the perpendicular is half the chord:



So, by the principle stated above, the square of the length of the perpendicular is $3 \times 2 = 6$; and the length of the perpendicular is $\sqrt{6}$ centimetres.

Like this, can't we draw lines of length as the square root of any number?

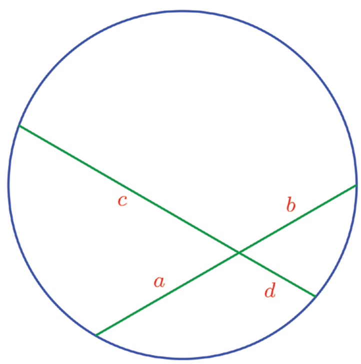
How do we draw a line of length $\sqrt{5}$ centimetres?

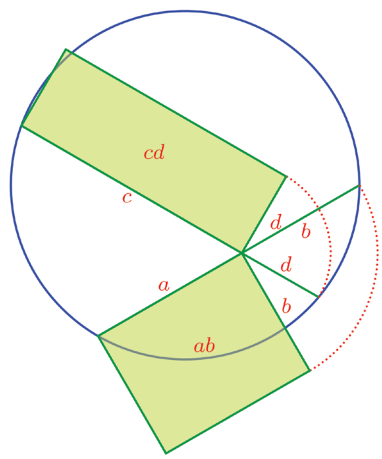
Can you draw a line of length $\sqrt{6}$ centimetres in any other way?

Areas

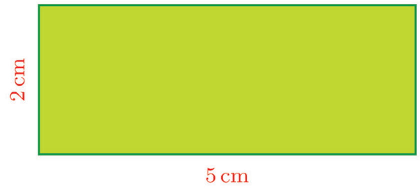
We have been talking about the products of the parts of two intersecting chords in a circle. The product of lengths can be considered as area. So the general principle about this can be stated like this:

> When two chords intersect within a circle, the rectangles with sides as the parts of each chord have equal areas





We can use this to change one side of a rectangle without affecting the area. For example, see this rectangle:

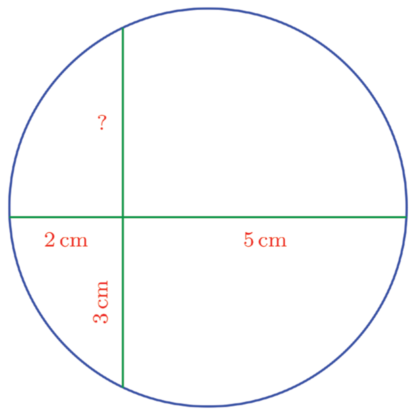


We want to draw a rectangle of the same area, with one side 3 centimetres.

The area of the rectangle is $5 \times 2 = 10$ square centimetres. If one side is to be 3 centimetres, the other should be $\frac{10}{3} = 3\frac{1}{3}$ centimetres. We cannot measure this exactly with a ruler, can we?

How can we use the result stated above?

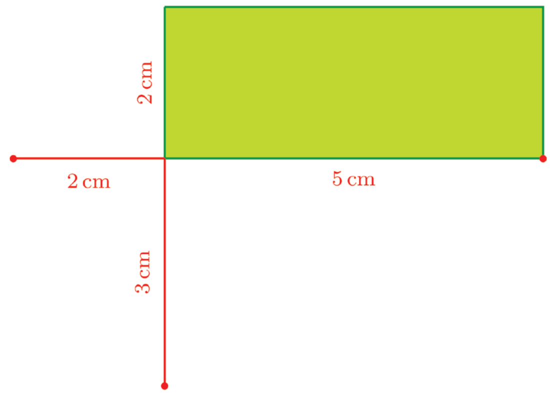
Imagine a pair of chords intersecting within a circle like this: the parts of one chord are 5 and 2 centimetres long, and one part of the other chord is 3 centimetres long. What would be the length of the other part?



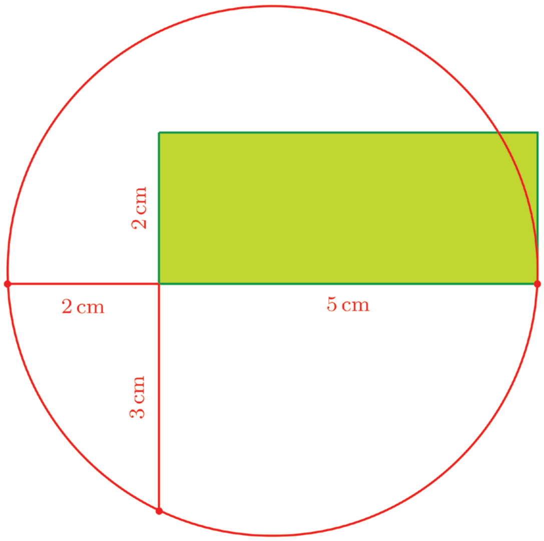
The length of this remaining part multiplied by 3 must be $2 \times 5 = 10$, right?

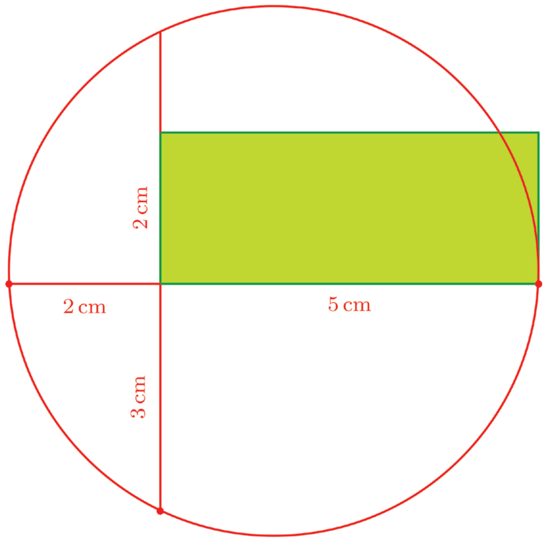
So this is the $3\frac{1}{3}$ centimetres long line we need. How do we actually draw it?

First let's extend the bottom side of the rectangle by 2 centimetres to the left and the left side downward by 3 centimetres:

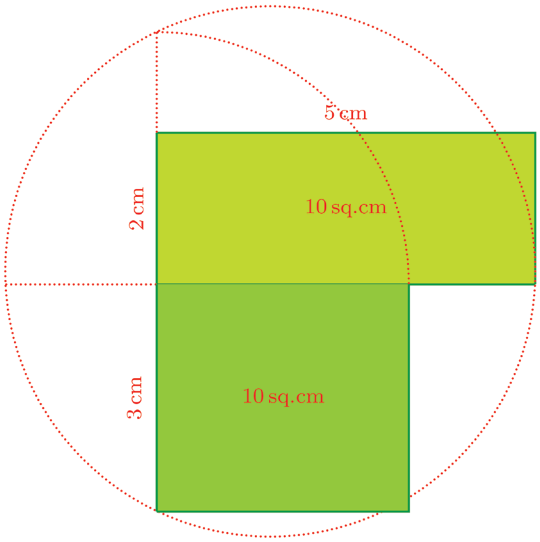


Next we draw the circle through the three points, left, right and down (we have seen how a circle can be drawn through three points in the section **Circumcircle** of the lesson **Circles** in the Class 8 textbook); and extend the left side of the rectangle to meet this circle.





Now the length we thus get can be marked horizontally to draw the rectangle we want:

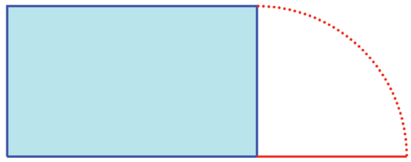


To draw such rectangles, we need not know the lengths of the sides of the original rectangle; only how much a side has to be lengthened or shortened.

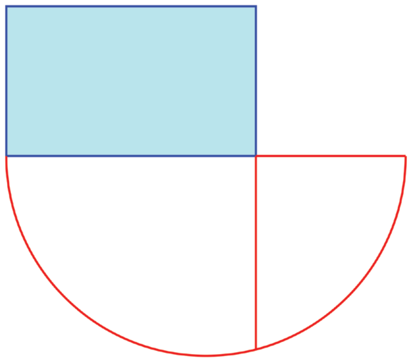
The corollary about the parts of a diameter cut by a perpendicular chord can be used to draw a square of the same area as a rectangle.

For example, see this rectangle:

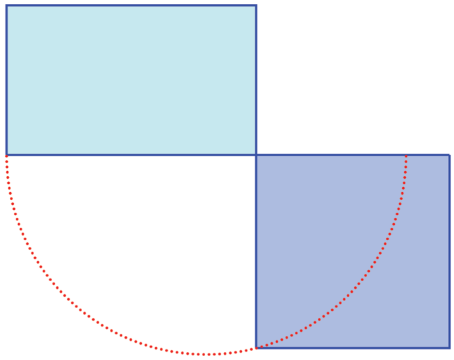
To draw a square of the same area as this, first extend the width of the rectangle by its height:



Now draw a semicircle downward with the bottom line as the diameter, and extend the right side of the rectangle downward to meet it:

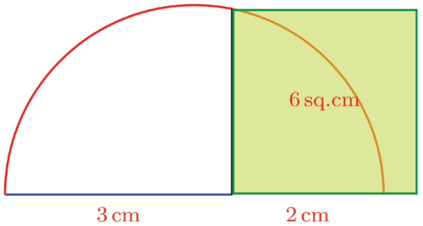


This line is the side of the square we want (why?)



Note that here also, we don't need the lengths of the sides of the rectangle.

We can also use this idea to draw a square of specified area. For example, we saw in the last section how we can draw a line of length $\sqrt{6}$ centimetres using this. With this as the length of a side, we can draw a square of area 6 square centimetres:



(1) Draw a rectangle of width 5 centimetres and height 4 centimetres.

(i) Draw a rectangle of the same area with width 6 centimetres.

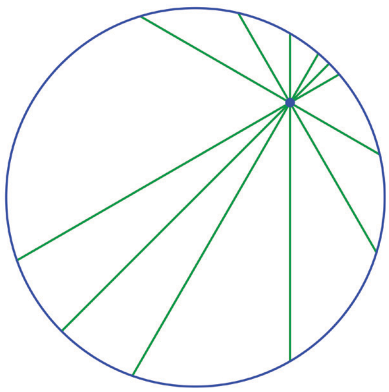
(ii) Draw a square of the same area.

(2) Draw a square of area 15 square centimetres.

(3) Draw a square of area 5 square centimetres in three different ways (Hint: Recall Pythagoras Theorem).

Line and point

See this picture:

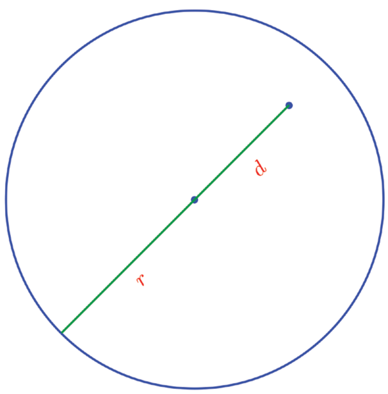


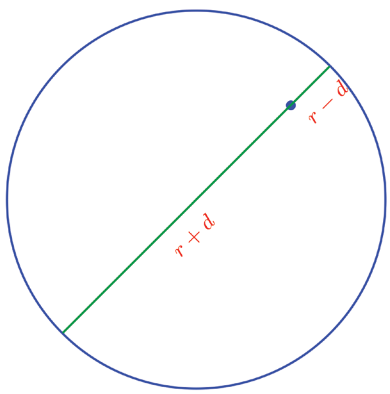
Draw a circle in GeoGebra and mark points A on the circle and P inside the circle. Select the Line tool and draw AP. Mark the point B where this line intersects the circle again. Hide the line and draw the chord AB. Compute PA x PB. Move A along the circle. Does the value of PA x PB change? What if we change the position of P? What happens to the product as we move P away from the centre of the circle? What if P is on the circle? When it is outside the circle?

Some chords pass through a single point within the circle. The point divides each chord into two parts. According to the general principle we have been using now, the product of the lengths of each pair of these parts gives the same number.

To calculate this number, let's draw the diameter through this point. Since it is also a chord of the circle, the product of the lengths of the parts into which the point divides the diameter is also equal to this number.

If we denote the radius of the circle as r and the distance of the point from the centre of the circle as d, we can write the lengths of the parts of the diameter like this:

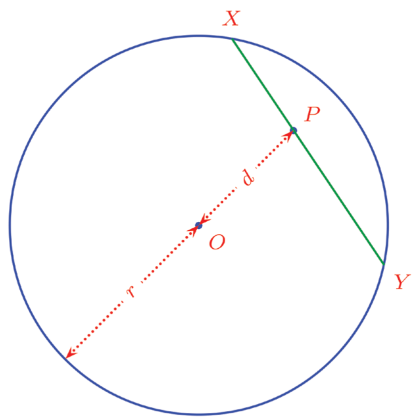




So, the product of these parts is

$$(r + d)(r - d) = r^2 - d^2$$

We have seen that the products of the lengths of the parts of any chord through the point is the same number. Thus the product of the lengths of the parts of every such chord is $r^2 - d^2$.



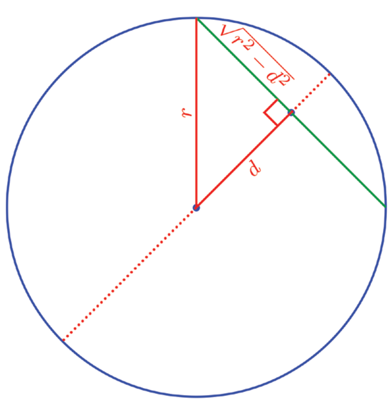
$$PX \times PY = r^2 - d^2$$

Thus we have this general result:

For any chord XY of a circle of radius r, passing through a point P within the circle at a distance d from the centre,

$$PX \times PY = r^2 - d^2$$

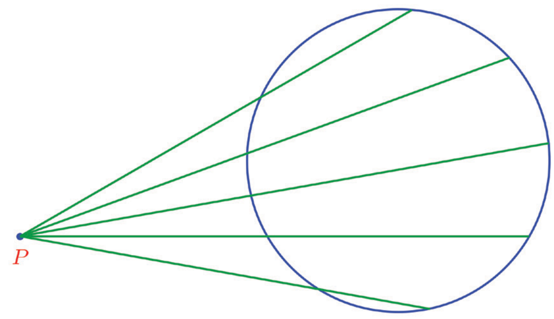
Geometrically, $r^2 - d^2$ is equal to the square of half the chord through P, perpendicular to the diameter through P.



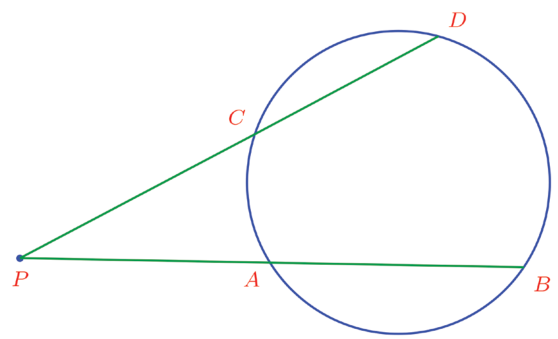
In the result above, P is a point within the circle and X, Y are points where a line through P intersects the circle, right? And PX, PY are the distances of these points from P.

What if we take a point outside the circle and draw lines intersecting the circle?

For such lines also, is the product of the distances from P to the points of intersection with the circle the same number?

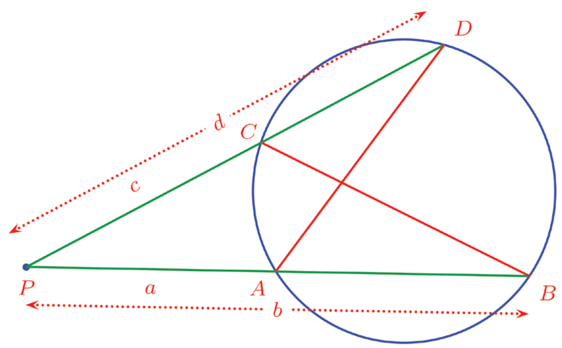


Let's take two such lines and check this:



We want to check whether the products $PA \times PB$ and $PC \times PD$ give the same number.

For that we draw lines to make triangles as in the picture below. We denote the lengths PA and PB as a and b, and the lengths PC and PD as c and d:



The triangle PAD and PCB have the same angle at P. The angles at B and D are equal (why?)

Thus these triangles have the same angles and so sides opposite to equal angles are scaled by the same factor.

In these triangles the lengths of the sides opposite to the angles at B and D are c and a; and the lengths of the sides opposite to the angles at C and A are b and d.

They are scaled by the same factor. Denote the scale factor as k, then we get

$$c = ka$$

$$b = kd$$

From this we get, as in the case of a point within the circle,

$$ab = cd$$

(Do it!)

That is,

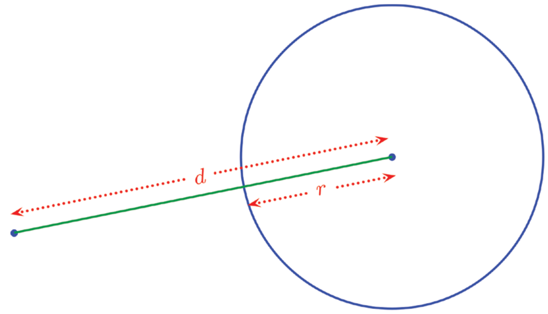
$$PA \times PB = PC \times PD$$

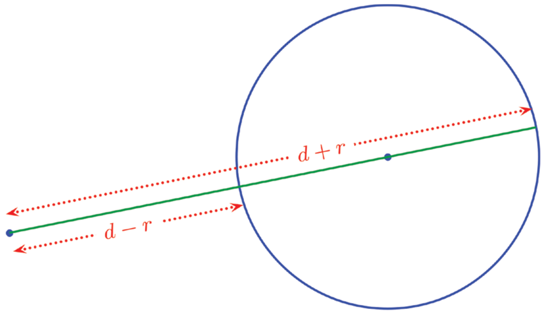
So, what do we see here?

> For all lines through a point outside a circle intersecting the circle at two points, the product of the distances from the point to the points of intersection with the circle is the same number.

Can we calculate this number also using the radius of the circle and the distance of the point from the centre of the circle?

Let's denote the radius of the circle as r and the distance of the point from the centre of the circle as d. We take the line joining the point to the centre of the circle to compute the product of the distances:

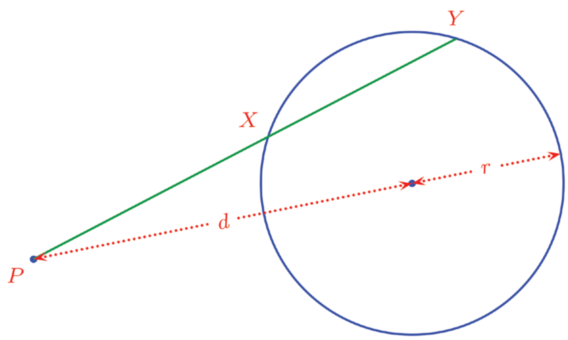




Here the distances from the point to the points of intersection with the circle can be seen as $d - r$ and $d + r$; and so their product is

$$(d + r)(d - r) = d^2 - r^2$$

Whatever line we draw through this point, the product of the distances from the point to the points of intersection with the circle would be this, isn't it?



$$PX \times PY = d^2 - r^2$$

Draw a circle in GeoGebra and mark a point P outside the circle. Draw the tangent from P to the circle and mark the point of contact T. Mark a point Y on the circle and draw the line PY. Mark the point X where this line meets the circle again. Compute PT^2 (for this type PT^2 as Input) and PX × PY. Compare these two numbers. Change the position of P and check.

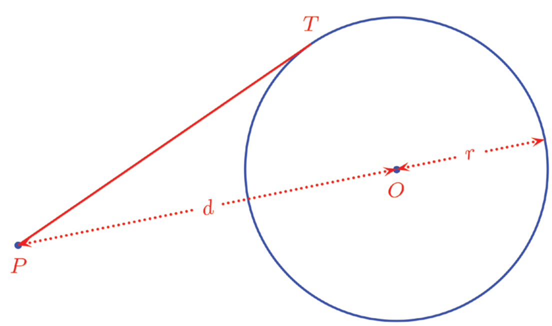
If a line from a point P outside a circle of radius r at a distance d from the centre cuts the circle at X and Y, then

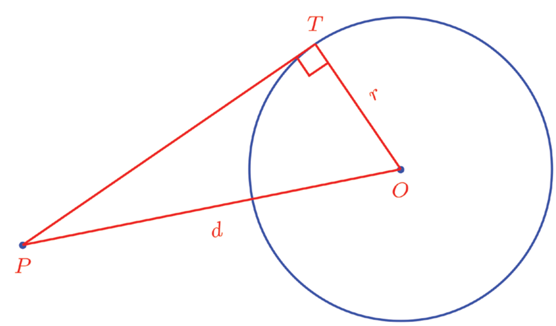
$$PX \times PY = d^2 - r^2$$

If P is a point inside the circle, we have seen that the product $PX \times PY$ can be geometrically seen as the square of half the length of a certain chord through P.

Is there such a geometric meaning if P is outside the circle?

To see this, draw the tangent from P to the circle and mark the point of contact as T. Joining T and P with the centre of the circle, we get a triangle:



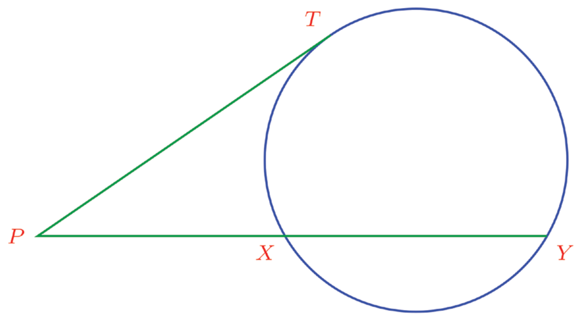


Since this is a right triangle (why?)

$$d^2 - r^2 = PT^2$$

Thus we have this result

Whatever line we draw from a point outside a circle, intersecting the circle at two points, the product of the distances from the point to the points of intersection is the square of the length of the tangent from the point.



$$PX \times PY = PT^2$$

Now let's compare the results about the lines drawn through a point inside or outside a circle

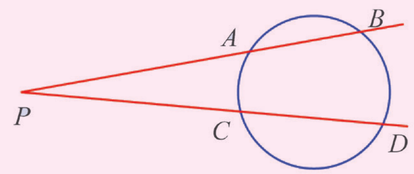
If we want to select a point at a distance d inside a circle of radius r, the distance d must be shorter than the radius r; and in this case the product of the distances from P to the points of intersection of a line through P with the circle is $r^2 - d^2$. If the point is to be outside the circle, d must be longer than the radius r; and the product as above is $d^2 - r^2$.

Whether the point is inside or outside the circle, what we do here is to subtract the larger of r^2 and d^2 from the smaller; thus in both cases we take the absolute value of $r^2 - d^2$ (the section **Distances** of the lesson **Real Numbers** in the Class 9 textbook).

So, we can say this in general:

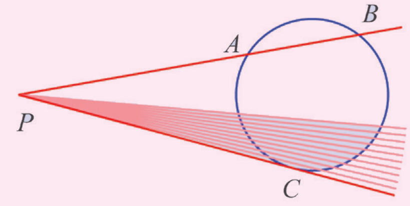
Unchanging relations

See this picture:



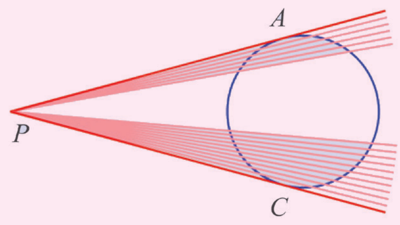
We know that $PA \times PB = PC \times PD$.

What if the bottom lines turns to become the tangent at C?



PD and PC coincide; and the relation between lengths becomes $PA \times PB = PC^2$

Suppose now the top line also turns to become the tangent at A:



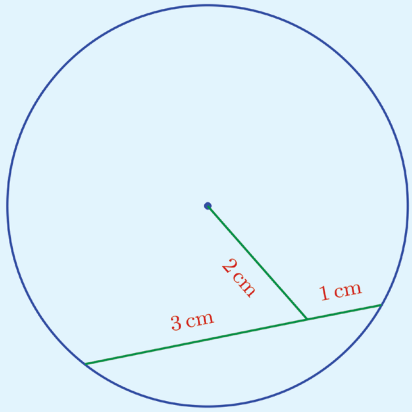
The relation becomes, $PA^2 = PC^2$; that is $PA = PC$.

We have already seen that the lengths of the tangents from a point are equal, haven't we?

If X and Y are the points of intersection of a circle of radius r with a line through a point P inside or outside the circle, at a distance d from the centre, then

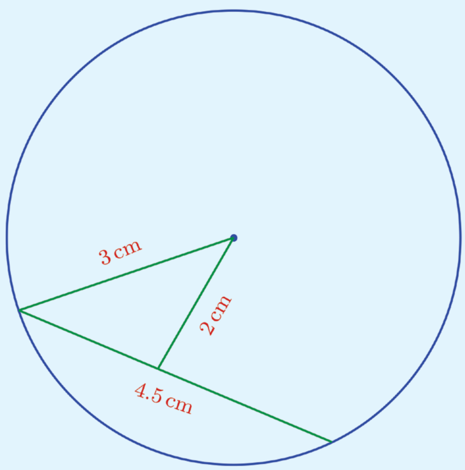
$$PX \times PY = |r^2 - d^2|$$

(1) In the picture, a line from the centre of a circle cuts a chord into two parts:



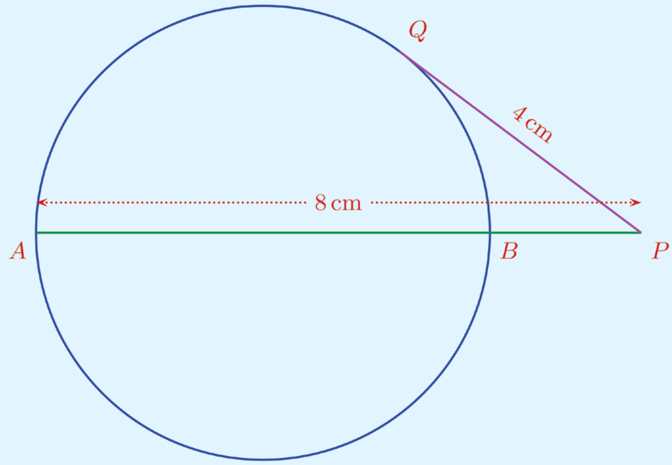
What is the radius of the circle?

(2) In the picture, a line from the centre of a circle meets a chord:



Find the lengths of the two parts of the chord.

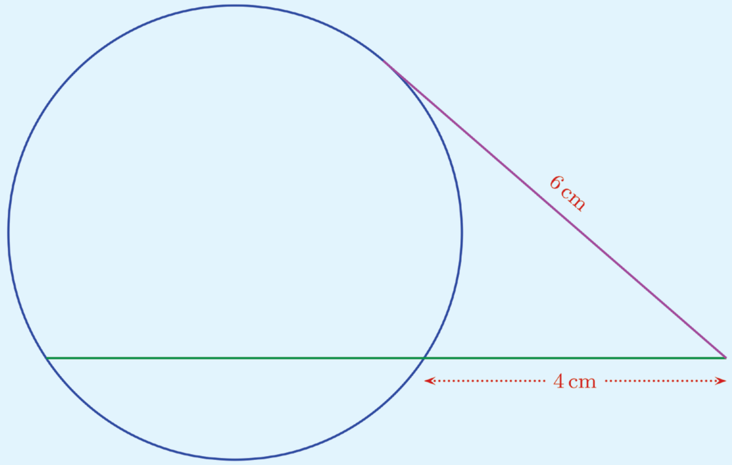
(3) In the picture, AB is a diameter of the circle and it is extended to a point P. The tangent from P touches the circle at Q:

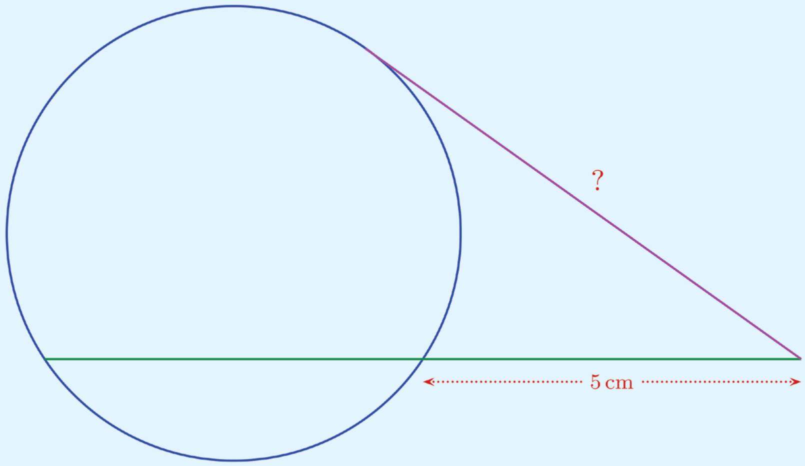


What is the radius of the circle?

(4) In the first of the two pictures below, the line joining two points of a circle is extended outward to a point and then a tangent is drawn from this point to the circle.

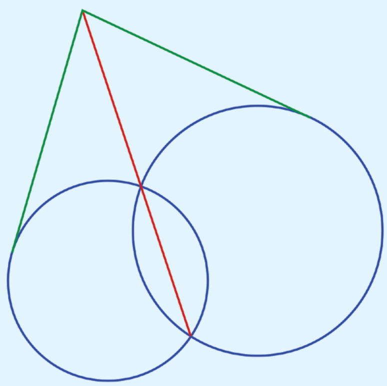
In the second picture, the same line is extended a bit more to a point and a tangent is drawn from this point:





What is the length of this tangent?

(5) In the picture, the line joining the points of intersection of two circles is drawn and from a point on it, one tangent is drawn to each circle:



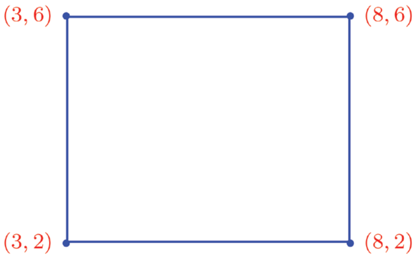
Prove that the lengths of these tangents are equal.

11 Geometry And Algebra

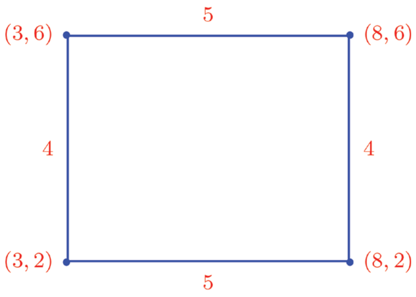
Parallelograms

In the chapter Coordinates, we have seen how points can be represented by pairs of numbers and did some problems using this idea.

For example, if in a rectangle with sides parallel to the axes, two of the opposite corners are (3, 2) and (8, 6), we can calculate the other two corners:

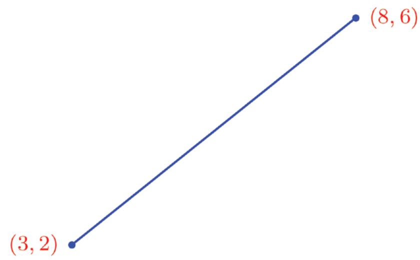


We can also calculate the lengths of the sides of the rectangle:



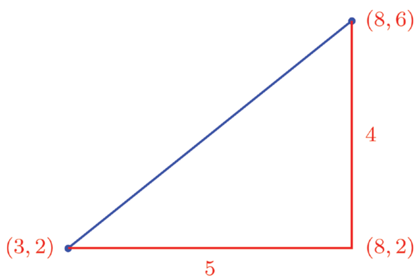
We can state this in a different way.

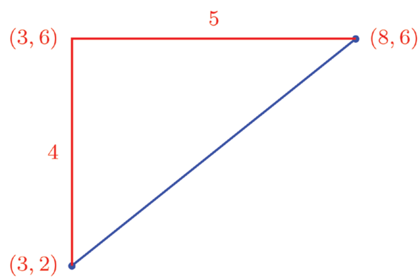
As we move from (3, 2) to (8, 6) what are the changes in the coordinates?



- The x-coordinate increases by $8 - 3 = 5$
- The y-coordinate increases by $6 - 2 = 4$

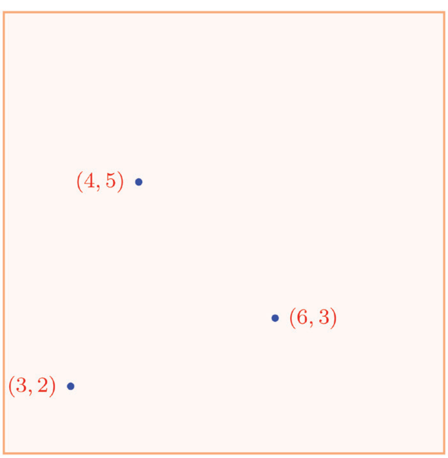
What if we make these changes one by one?

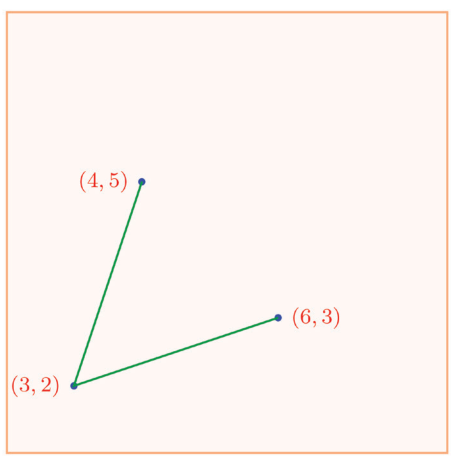


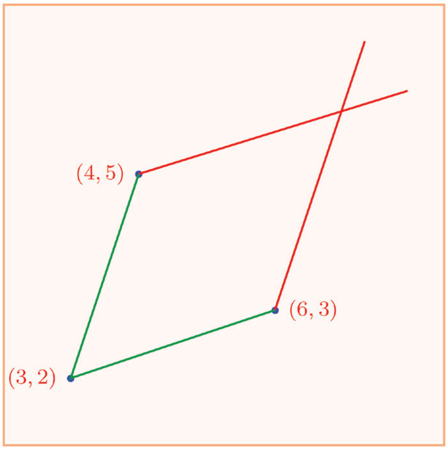


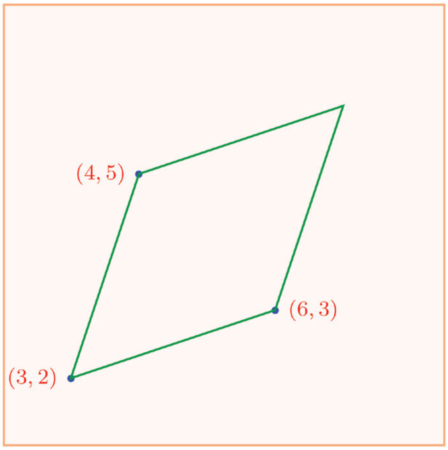
The changes in the coordinates when we move from one point to another can be shown like this as sides of a right triangle with its perpendicular sides parallel to the axes.

Now look at these pictures:









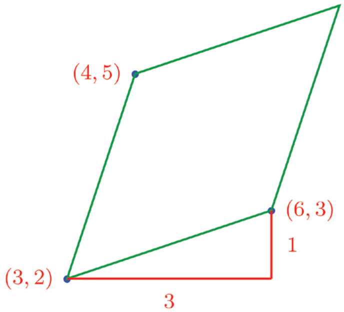
A parallelogram is drawn with three specific points as vertices.

What are the coordinates of its fourth vertex?

For that let's see what the changes in coordinates are when we move from the bottom-left to the bottom-right vertex.

- The x-coordinate increases by $6 - 3 = 3$.
- The y-coordinate increases by $3 - 2 = 1$.

As noted above, we can show these changes by drawing a right triangle:

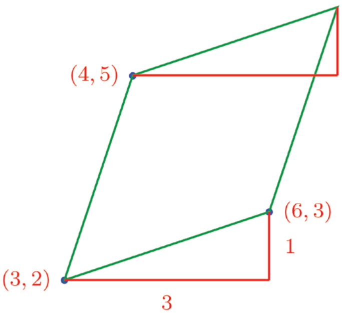



In GeoGebra, mark the points A(0, 0) and two other points B and C. Typing

(x(B) + x (C), y(B) + y (C))

as Input creates the point D. Draw the quadrilateral ABDC. Isn't it a parallelogram? Why? Change the positions of B and C and check.

We want to calculate the coordinates of the top-right vertex; for that we need only know the changes in the top-left vertex. For that, we draw a triangle at the top, like the one at the bottom:



The changes in the coordinates are the lengths of the perpendicular sides of this triangle. How do we calculate them?

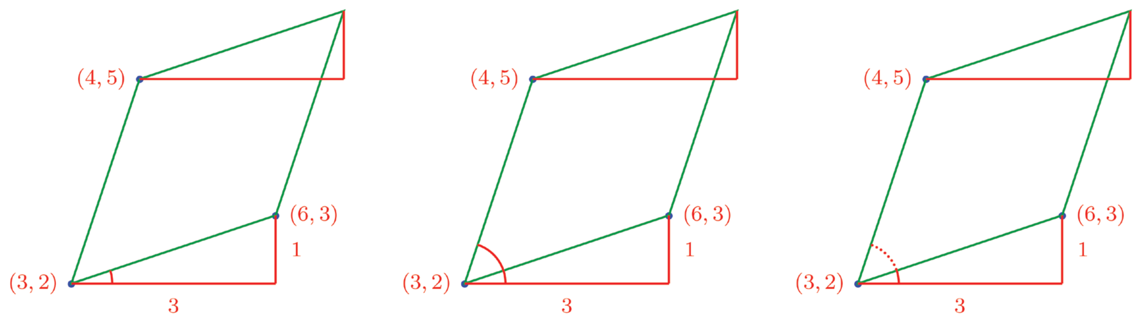
Doesn't this triangle look equal to the bottom triangle? How do we check whether this is correct?

The hypotenuses of the two triangles are opposite sides of a parallelogram; and so they are equal.

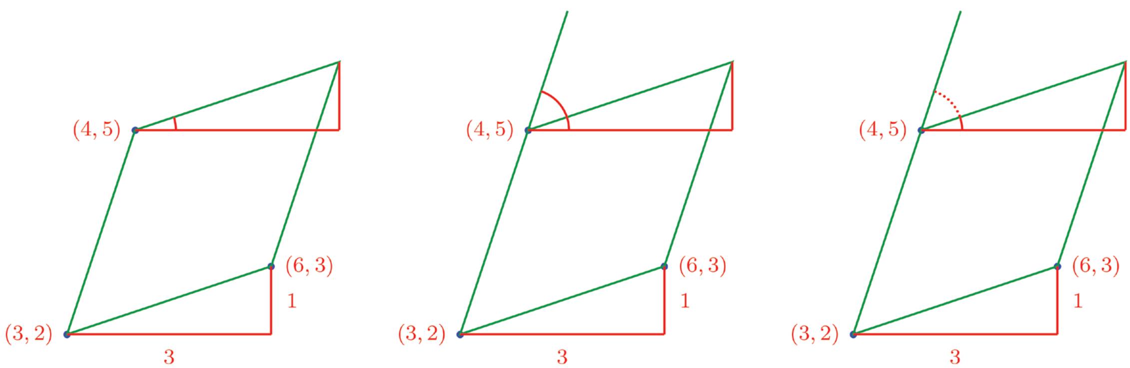
What about the angles of these triangles?

We can see the angle at the left corner of the bottom triangle as the difference of two angles:

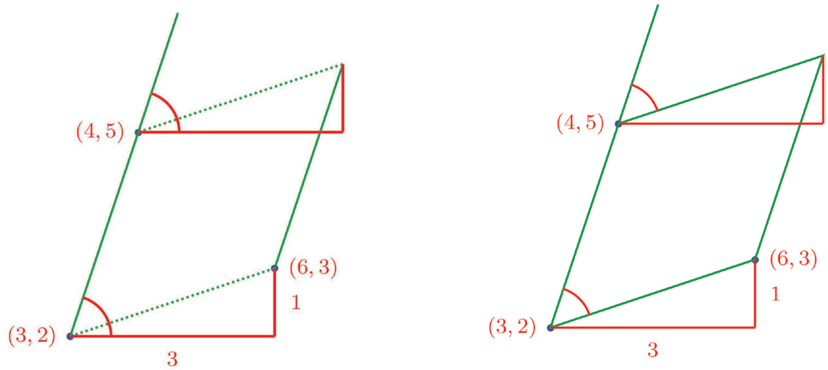
> Draw triangle ABC in GeoGebra and mark the coordinates of its vertices. Use the Move tool to shift the triangle right by 3. What happens to the coordinates of the vertices? Now shift this triangle upward by 2. What are the coordinates now? What are the relations between the coordinates of the vertices of the first triangle and the last triangles? Try this on a parallelogram instead of a triangle.



The left angle of the top triangle can also be seen as the difference of two angles like this:



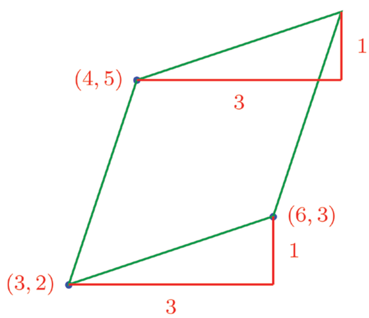
Look at the angles which are subtracted at the top and bottom:



The angles in the first picture are two small angles made by the green line intersecting the red parallel lines, and so they are equal. The angles in the second picture are the two small angles made by the same line intersecting the green parallel lines at the top and bottom, and so they are also equal. (The section **Matching angles** of the lesson **Parallel Lines** in the Class 7 textbook).

So the differences of these angles are also equal. That is in our right triangles, the angles at the bottom are equal, and so the third angles are also equal. Since the hypotenuses and the angles on its ends are equal, the other two sides of the triangles are also equal (The section **Two angles** of the lesson **Equal Triangles** in the Class 8 textbook).

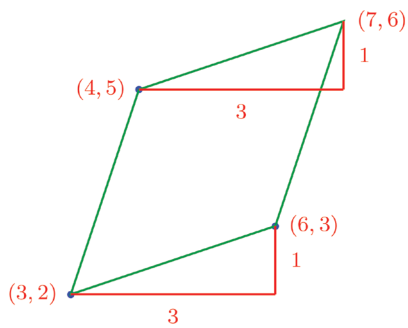
Thus we can mark the lengths of the perpendicular sides of the top-right triangle also like this:



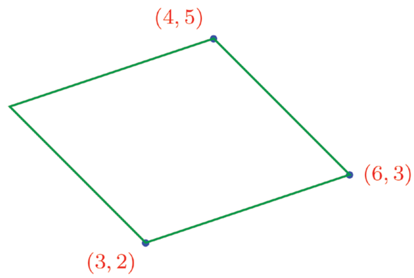
To move a figure drawn in GeoGebra to a specific position, we can use the Translate by Vector tool. Suppose we want to shift a parallelogram we have drawn 3 right and 2 up. For this, mark the points (0, 0) and (3, 2). Select the Translate by Vector and click on the parallelogram first and then the points (0, 0) and (3, 2). What happens to the coordinates of the parallelogram? Check using a point other than (3, 2).

Now can't we calculate the coordinates of the fourth vertex of the parallelogram?

- The x-coordinate $4 + 3 = 7$.
- The y-coordinate $5 + 1 = 6$.

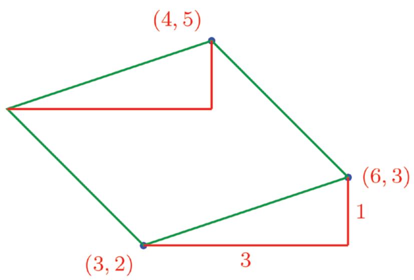


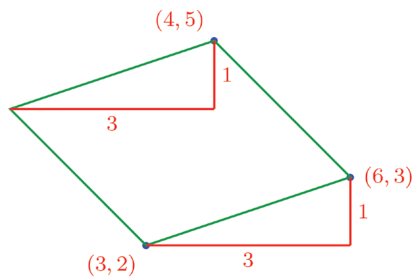
With the same three points we started with in this problem, can't we draw another parallelogram?



What is the fourth vertex of this parallelogram?

As we did earlier, we can find it by drawing right triangles at the top and bottom, with perpendicular sides parallel to the axes:





Here, to reach the corner from (4, 5), we must move downward by 1 and then move left by 3.

That is,

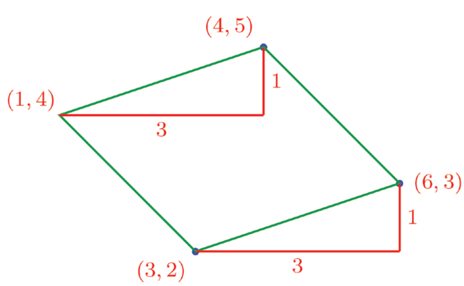
- Decrease the y-coordinate by 1.
- Decrease the x-coordinate by 3.

So, the coordinates of the fourth vertex are (1, 4).

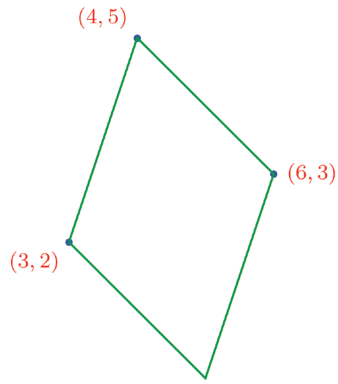
In GeoGebra, mark three points A, B, C. If we type

(x(B) + x(C) − x(A), y(B) + y(C) − y(A))

We get a new point D. Is ABDC a parallelogram? Why? Change the positions of the points.

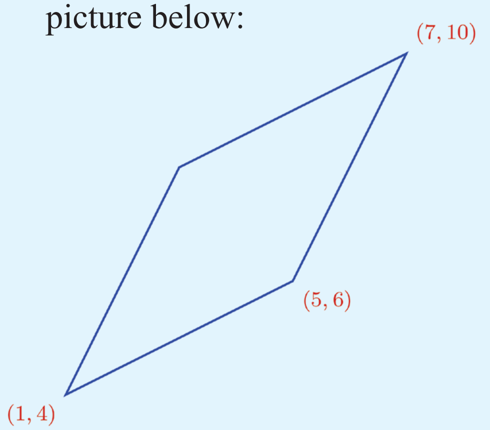


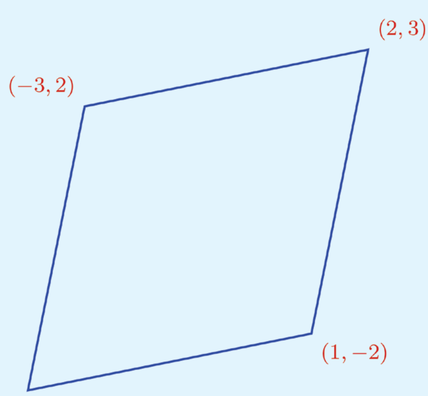
Can't we draw a third parallelogram with the same three points as vertices?



Can you compute the coordinates of its fourth vertex?

(1) Calculate the coordinates of the fourth vertex of the parallelogram in each picture below:

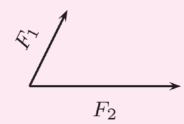




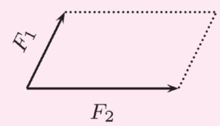
Force parallelogram

We can produce the same effect of two forces acting along different directions on a body, by a single force acting along a definite direction.

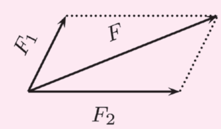
There is a method, recognized through experiments, to find this force and its direction. Draw two lines from a point with their lengths proportional to the forces (such as for example one centimetre for one newton), along the directions of the forces:



Next draw a parallelogram with these as adjacent sides:

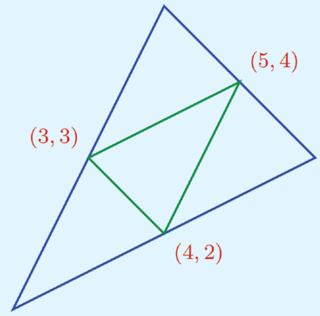


The single force to replace these two forces acts along the diagonal of this parallelogram; and its magnitude is the length of this diagonal, in the scale chosen.



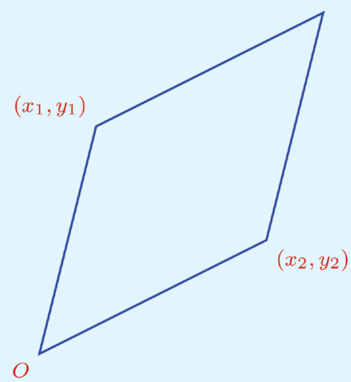
This is known as the Parallelogram Law of Forces.

(2) The sides of the larger triangle in the picture are parallel to the sides of the smaller triangle.

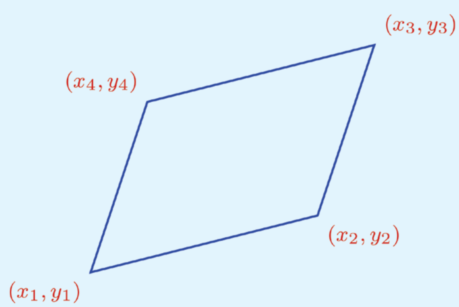


Calculate the coordinates of the vertices of the large triangle.

(3) The adjacent sides of a parallelogram are the lines joining the origin to the points with coordinates (x_1, y_1) and (x_2, y_2). What are the coordinates of the fourth vertex?



(4) In the picture, the coordinates of the four vertices of a parallelogram are marked:



Prove that the coordinates are connected by the relations below:

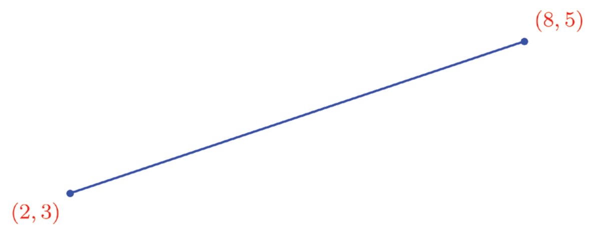
$$x_1 + x_3 = x_2 + x_4$$

$$y_1 + y_3 = y_2 + y_4$$

Midpoint

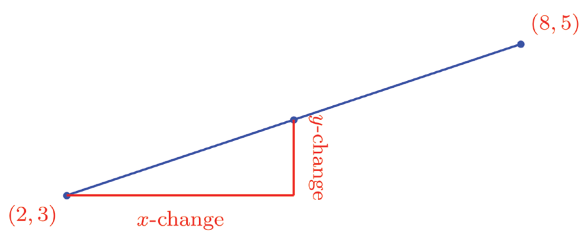
What we have done in the problems so far is to calculate the coordinates of some points from the coordinates of some other points geometrically related to it. Let's look at another problem of this kind.

The picture shows the coordinates of two points at the ends of a line:

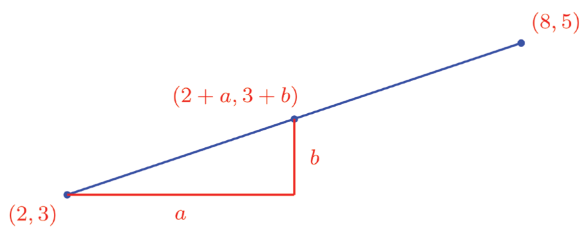


We want to find the coordinates of the midpoint of this line.

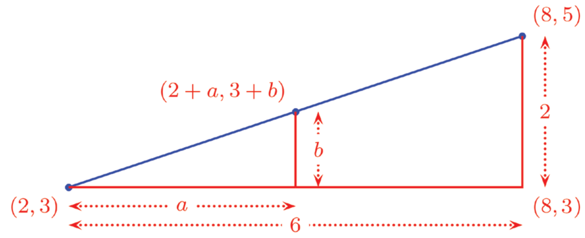
What we want to find is the point at half the distance between (2, 3) and (8, 5). For this, we have to find the changes in the *x*-coordinate and the *y*-coordinate as we move from (2, 3) to this point:



If we denote these by *a* and *b*, we can write the coordinates of the midpoint as $(2 + a, 3 + b)$:

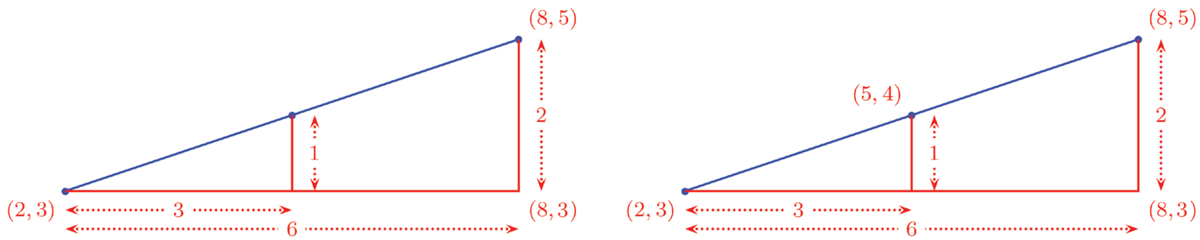


We can show the changes in coordinates as we move from (2, 3) to (8, 5) also, by drawing a triangle:

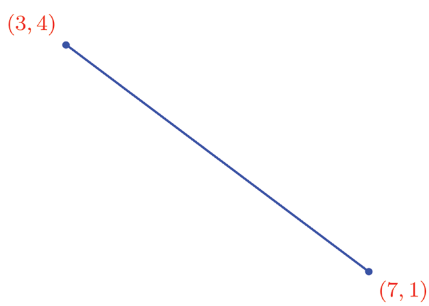


The small and large right triangles in this picture have the same angles (how?). So, the lengths of their sides are scaled by the same factor.

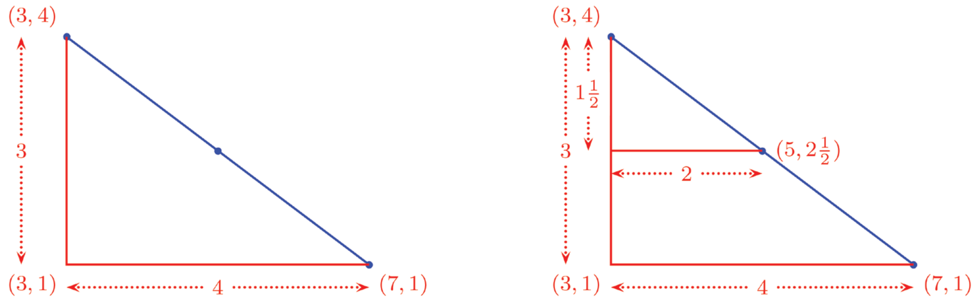
The hypotenuse of the smaller triangle is half that of the larger. So, their perpendicular sides are also halved. Thus a and b are 3 and 1. The midpoint is thus (5, 4):



What if the line is like this?



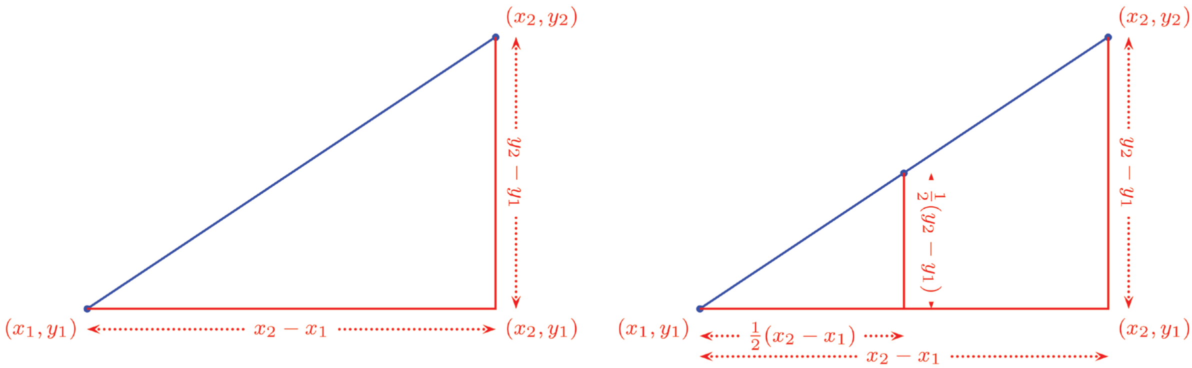
As we move from (3, 4) to (7, 1), the x-coordinate is increased by 4 and the y-coordinate is decreased by 3. So to reach the midpoint, the x-coordinate must be increased by 2 and the y-coordinate must be decreased by $1\frac{1}{2}$.



Thus we can calculate the coordinates of the midpoint as $\left(5, 2\frac{1}{2}\right)$.

Now let's take the coordinates of two points as (x_1, y_1), (x_2, y_2) in general and calculate the midpoint of the line joining them.

They may be positioned in different ways. First consider this:



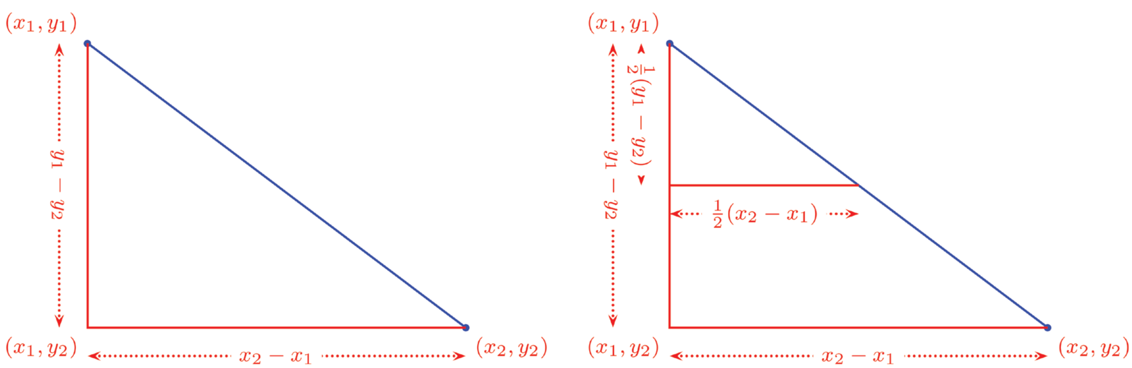
The x-coordinate of the midpoint is

$$x_1 + \frac{1}{2}(x_2 - x_1) = \frac{1}{2}(x_1 + x_2)$$

and the y-coordinate is

$$y_1 + \frac{1}{2}(y_2 - y_1) = \frac{1}{2}(y_1 + y_2)$$

What if the points are like this?



The x-coordinate of the midpoint is

$$x_1 + \frac{1}{2}(x_2 - x_1) = \frac{1}{2}(x_1 + x_2)$$

and the y-coordinate is

$$y_1 - \frac{1}{2}(y_1 - y_2) = \frac{1}{2}(y_1 + y_2)$$

To get the midpoint of the line joining the points A and B in GeoGebra, just type (A+B)/2 as the Input.

Draw another picture interchanging the position of (x_1, y_1) and (x_2, y_2) and see. What do we get?

The midpoint of the line joining (x_1, y_1) and (x_2, y_2) is

$$\left(\frac{1}{2}(x_1 + x_2), \frac{1}{2}(y_1 + y_2)\right)$$

(1) A circle is drawn with the line joining (2, 3) and (6, 5) as diameter. What are the coordinates of the centre of the circle?

(2) The coordinates of two opposite vertices of a parallelogram are (4, 5) and (1, 3). Calculate the coordinates of the point of intersection of the diagonals. What are the coordinates of the midpoint of the other diagonal?

(3) A quadrilateral is drawn with vertices $A(1, 3)$, $B(8, 6)$, $C(12, 13)$, $D(5, 10)$. Prove that $ABCD$ is a parallelogram.

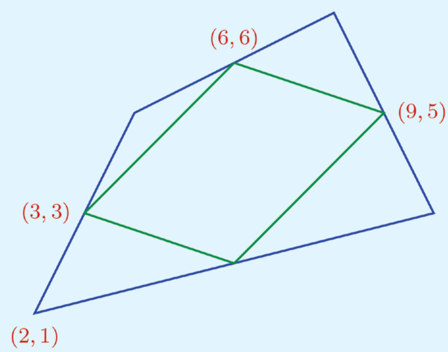
(4) Prove that the triangle with vertices (3, 5), (9, 13), (10, 6) is isosceles. Calculate its area.

(5) The centre of a circle is (1, 2) and a point on it is (3, 5). Find the coordinates of the other end of the diameter through this point.

(6) In the picture, the midpoints of the sides of a quadrilateral are joined to make a smaller quadrilateral within it:

(i) Calculate the coordinates of the other three vertices of the larger quadrilateral.

(ii) Calculate the coordinates of the fourth vertex of the smaller quadrilateral.

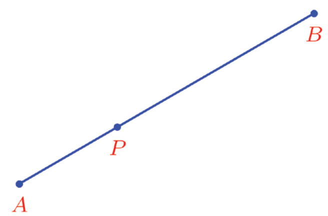


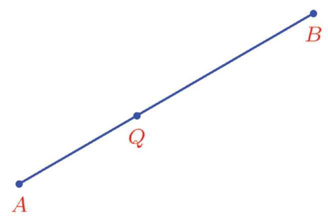
(7) Calculate the coordinates of the circumcentre of the triangle with vertices (0, 0), (0, 4), (3, 0).

Ratio

The midpoint of a line joining two points divides the line into two equal parts. Any other point on the line divides into parts of different lengths. We use ratios to specify the positions of such points (Recall the section **Parts of a line** of the lesson **Ratio** in the Class 8 textbook).

For example, see these pictures:





In the first picture, the length of AP is $\frac{1}{3}$ of the length of AB and the length of PB is the remaining $\frac{2}{3}$ of the length of AB.

How do we say this using a ratio?

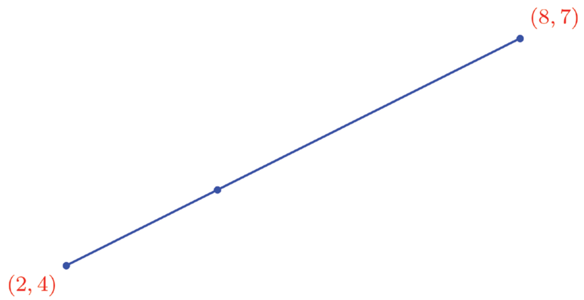
The point P divides AB in the ratio 1 : 2.

On the other hand what does it mean, when we say that in the second picture, the point Q divides AB in the ratio 2 : 3?

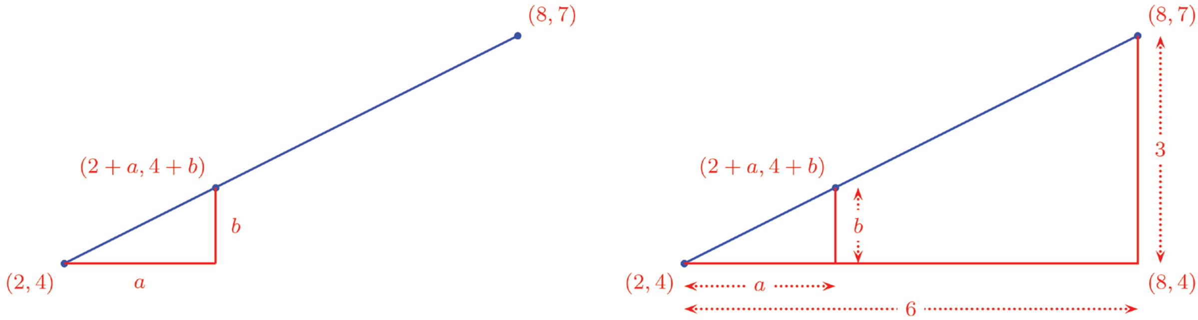
The length of AQ is $\frac{2}{5}$ of the length of AB (and the length of QB is the remaining $\frac{3}{5}$ of AB).

To find the coordinates of a point dividing a line in a specified ratio, we can use the same technique we used to find the midpoint (with some minor modifications).

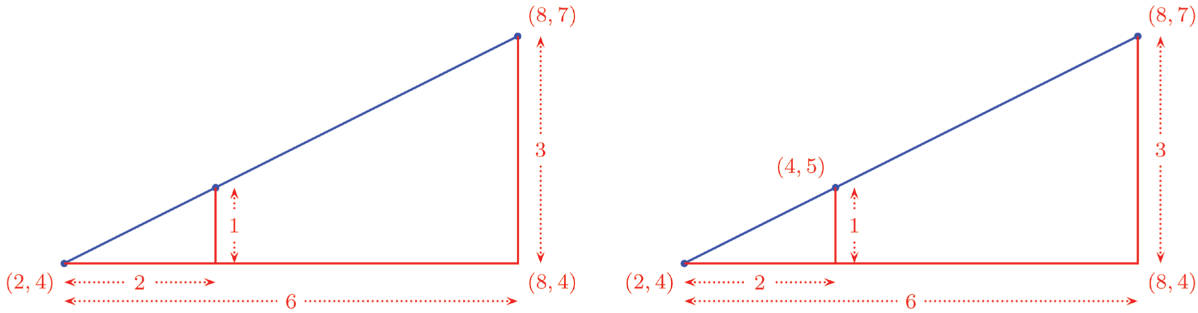
For example, let's calculate the coordinates of the point which divides the line joining (2, 4) and (8, 7) in the ratio 1 : 2



As in the case of midpoint, let's denote the change in the coordinates as we move from (2, 4) to the point we want as a and b and mark them on the triangle showing these changes; also mark the changes in coordinates as we move from (2, 4) to (8, 7):



Here the hypotenuse of the smaller triangle is $\frac{1}{3}$ of the hypotenuse of the larger triangle. So, the perpendicular sides are also scaled by the same factor. Using this, we can calculate their lengths and from that the coordinates of the point we want:



Denoting the points (2, 4) and (8, 7) as A and B, and the point we want as P, the computations done above can be written like this:

(1) Since $AP : PB = 1 : 2$, the length of AP is $\frac{1}{3}$ of the length of AB.

(2) As we move from A to B

- (i) the x-coordinate increases by $8 - 2 = 6$.
- (ii) the y-coordinate increases by $7 - 4 = 3$.

(3) The changes in coordinates as we move from A to P is $\frac{1}{3}$ of these

- (i) the x-coordinate increases by $\frac{1}{3} \times 6 = 2$.
- (ii) the y-coordinate increases by $\frac{1}{3} \times 3 = 1$.

(4) Coordinates of P

(i) x-coordinate is $2 + 2 = 4$.

(ii) y-coordinate is $4 + 1 = 5$.

(5) Coordinates of P are $(4, 5)$

Even if the coordinates of A and B and the ratio of division are different, we can use the same technique to compute the coordinates of the dividing point.

For example, how do we calculate the coordinates of the points which divides the line joining $(1, 6)$ and $(11, 2)$ in the ratio $3 : 5$?

As we did above, we can start by denoting the end points as A, B and the dividing point as P:

(1) Since $AP : PB = 3 : 5$, the length of AP is $\frac{3}{8}$ of the length of AB.

(2) As we move from A to B,

(i) the x-coordinate increases by $11 - 1 = 10$.

(ii) the y-coordinate decreases by $6 - 2 = 4$.

(3) The changes in coordinates as we move from A to P is $\frac{3}{8}$ of these

(i) the x-coordinate increases by $\frac{3}{8} \times 10 = 3\frac{3}{4}$.

(ii) the y-coordinate decreases by $\frac{3}{8} \times 4 = 1\frac{1}{2}$.

> In GeoGebra, to get the point dividing the line joining A and B in the ratio 3 : 5, just type A+(3/8)(B-A) as the Input.

(4) Coordinates of P.

(i) x-coordinate is $1 + 3\frac{3}{4} = 4\frac{3}{4}$

(ii) y-coordinate is $6 - 1\frac{1}{2} = 4\frac{1}{2}$

(5) Coordinates of P are $\left(4\frac{3}{4}, 4\frac{1}{2}\right)$

(1) The coordinates of the point A is $(3, 2)$ and the coordinates of the point B is $(8, 7)$. Calculate the coordinates of the points P and Q which divide AB in the ratios given below:

(i) $AP : PB = 2 : 3$.

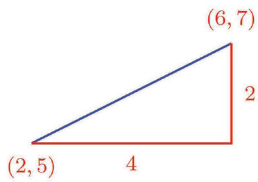
(ii) $AQ : QB = 3 : 2$.

(2) Calculate the coordinates of the points which divide the line joining (1, 6) and (5, 2) into three equal parts.

(3) The vertices of a triangle are (−1, 5), (3, 7), (1, 1). Find the coordinates of its centroid.

Line maths

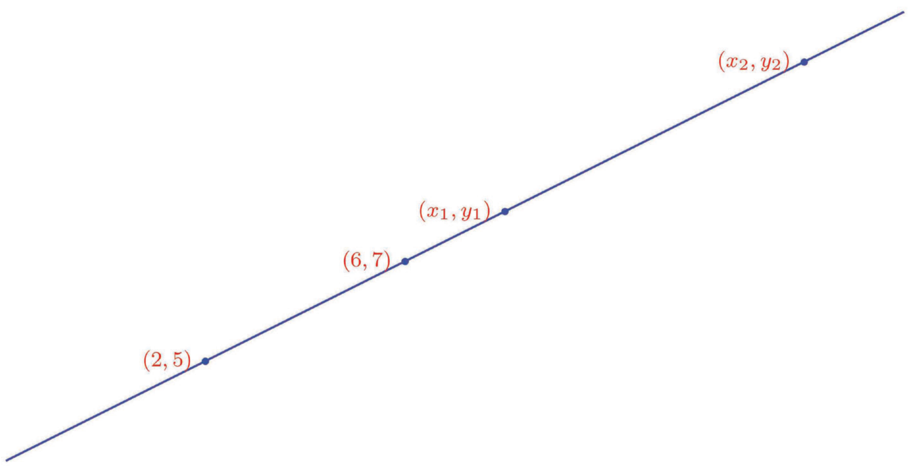
See this picture:



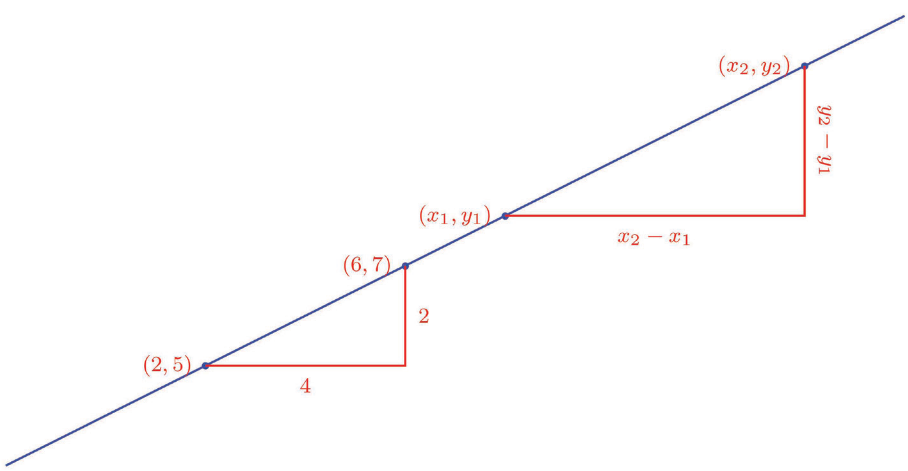
As we move from the point (2, 5) to (6, 7), the x-coordinate increases by 4 and the y-coordinate increases by 2.

That is, the increase in the y-coordinate is half the increase in the x-coordinate.

Let's extend this line and take two other points (x_1, y_1) and (x_2, y_2) on it:



The change in the coordinates as we move from (x_1, y_1) to (x_2, y_2) can be seen like this:



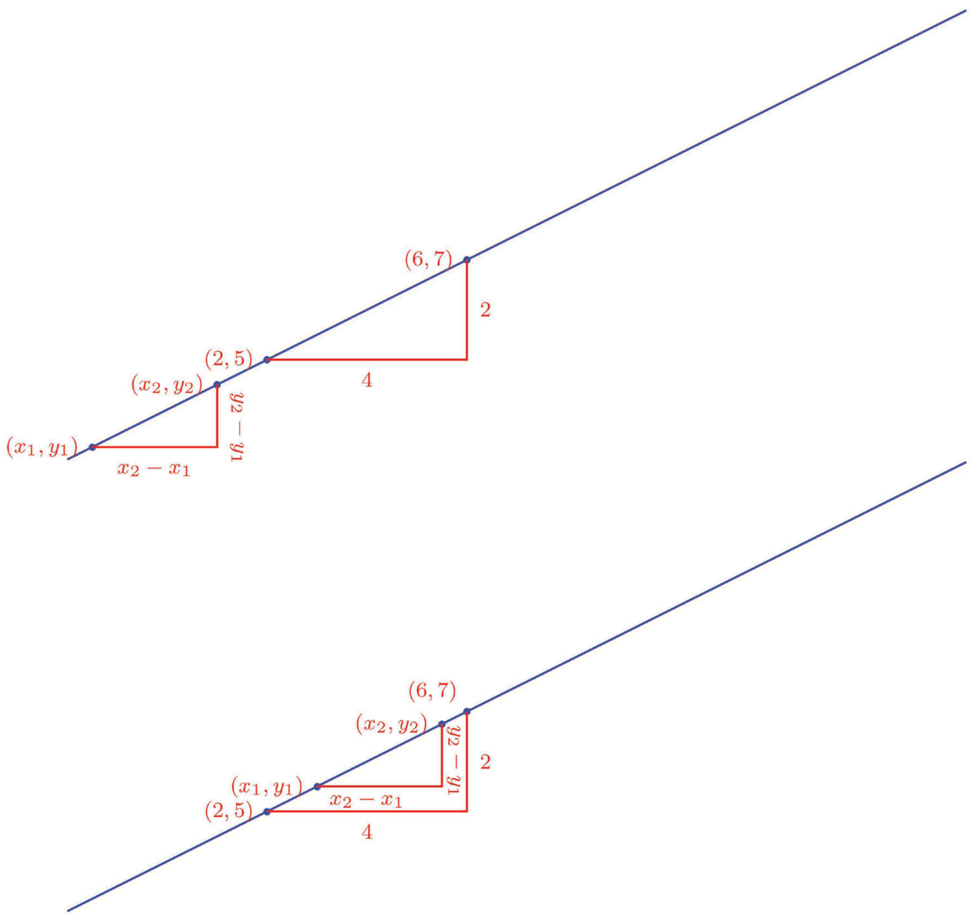
What is the relation between $y_2 - y_1$ and $x_2 - x_1$?

It is the same as the relation between the height and base of the smaller right triangle, isn't it? (Why?)

Thus we have

$$y_2 - y_1 = \frac{1}{2}(x_2 - x_1)$$

This is true wherever we take the points (x_1, y_1), (x_2, y_2) on the line joining (2, 5) and (6, 7), isn't it?

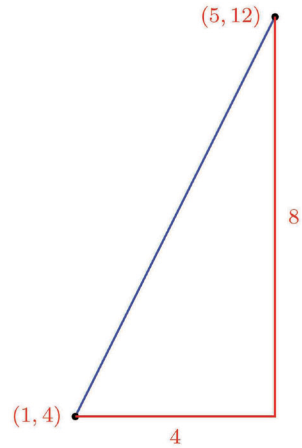


So, what can we say about the coordinates of any two points on this line?

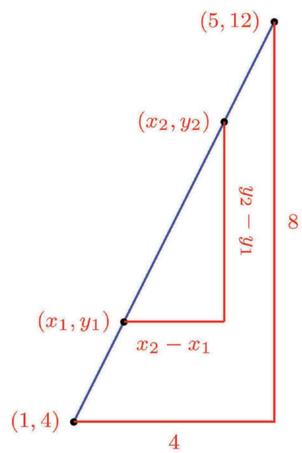
> On the line joining (2, 5) and (6, 7), as we move from any point to another, the change in the *y*-coordinates is half the change in the *x*-coordinate.

What if we take another pair of points instead of (2, 5) and (6, 7)?

For example, let's take (1, 4) and (5, 12). As we move from (1, 4) to (5, 12) along this line, the x-coordinate increases by 4 and the y-coordinate increases by 8; that is the y-difference is twice the x-difference:



And if we take two other points on this line?



Since the height of the larger triangle is twice the base, the height of the smaller triangle must also be twice the base. Thus

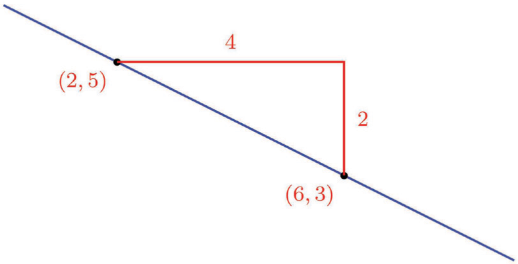
$$y_2 - y_1 = 2(x_2 - x_1)$$

In general,

On the line joining (1, 4) and (5, 12), as we move from any point to another, the change in the y-coordinates is twice the change in the x-coordinate.

In both these examples, as x increases, so does y. The opposite can also happen.

For example, taking the points (2, 5) and (6, 3) we see that as the x-coordinate increases by 4, the y-coordinate decreases by 2.



Make a slider a and mark a point A in GeoGebra. If we type (x(A)+a,y(A)+2a) as Input, we get a new point. Set Animation On for the slider and see the path of the point as it moves. This can be seen by setting Trace On for the point. Change the position of A and check.

This can be stated in a different way.

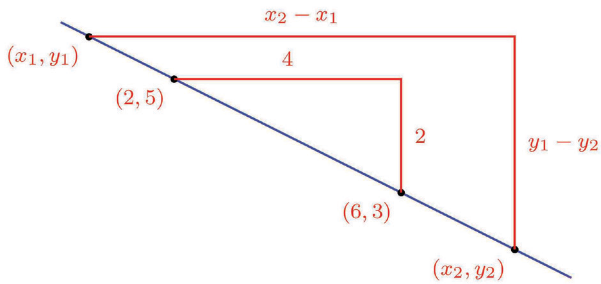
As we move from (2, 5) to (6, 3),

$$\text{change in } x\text{-coordinate is } 6 - 2 = 4$$

$$\text{change in } y\text{-coordinate is } 3 - 5 = -2$$

So we can say that the change in the y-coordinate is $-\frac{1}{2}$ times the change in the x-coordinate.

What if we take some other points (x_1, y_1) and (x_2, y_2) on this line?



In the picture, the horizontal side of the larger triangle is $x_2 - x_1$ and the vertical side is $y_1 - y_2$; So,

$$y_1 - y_2 = \frac{1}{2}(x_2 - x_1)$$

As we move from (x_1, y_1) to (x_2, y_2)

$$\text{change in } x\text{-coordinate is } x_2 - x_1$$

$$\text{change in } y\text{-coordinate is } y_2 - y_1$$

So the above equation can be rewritten as the relation between coordinate changes like this:

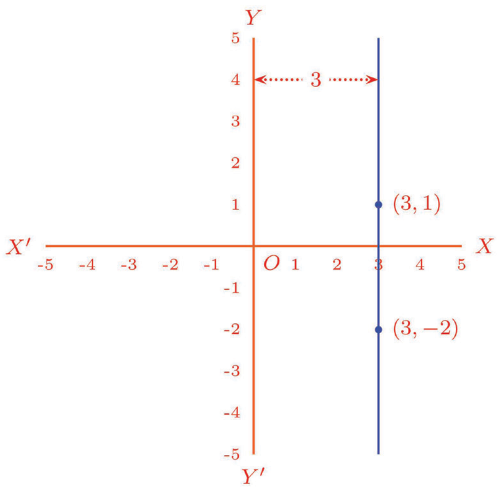
$$y_2 - y_1 = -\frac{1}{2}(x_2 - x_1)$$

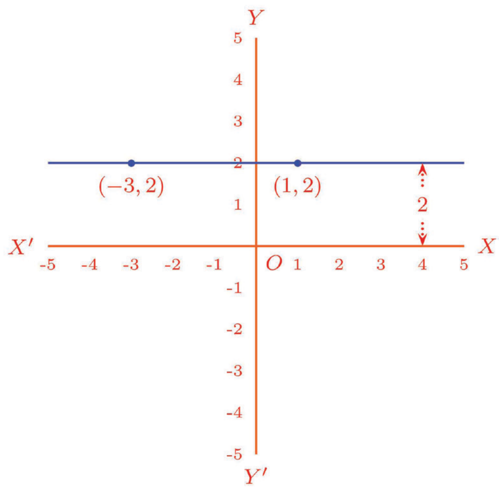
What can we say in general about the changes in coordinates?

On the line joining (2, 5) and (6, 3), as we move from any point to another, the change in the y-coordinates is $-\frac{1}{2}$ times the change in the x-coordinates.

In all these examples, as we move from one point to another on a line, both the x-coordinates and the y-coordinates change. Are there lines for which this is not so?

On a line parallel to the y-axis, all points have the same x-coordinate, right? On the other hand on a line parallel to the x-axis, all points have the same y-coordinate:





So, the general result we get from all these examples can be said like this:

> For any two points on a line, not parallel to either axis, the change in the y-coordinate is the change in the x-coordinate multiplied by a fixed number

This can be used to find the coordinates of the points on a line, if we know the coordinates of any two points on the line.

For example, let's take the line in the first example above. We have seen that for any two points on the line joining the points (2, 5) and (6, 7), the change in the y-coordinate is half the change in the x-coordinate.

So, if we take one point on this line and add 2 to the x-coordinate and 1 to the y-coordinate, we get another point on this line. Thus from the point (2, 5) on the line, we get the point (4, 6) on this line.

In the same way, from (4, 6) on the line we get the point (6, 7) given in the problem.

Generally by repeatedly adding 2 to 2 and adding 1 to 5, we get these points on the line:

(4,6), (6,7), (8,8), ...

Instead of adding if we subtract, we get these points.

(0, 4), (−2, 3), (−4, 2), ...

How do we do this problem like this?

Find the coordinates of some other points on the line joining (4, 6) and (7, 4).

As we move from (4, 6) to (7, 4),

the x-coordinate increases by 3

the y-coordinate decreases by 2

So, starting with (4, 6) if we repeatedly add 3 to the x-coordinate and subtract 2 from the y-coordinate, we get several points on the line such as

(7, 4), (10, 2), (13, 0), ...

If we repeatedly subtract 3 from the x-coordinate and add 2 to the y-coordinate we get another set of points on the line:

(1, 8), (−2, 10), (−5, 12), ...

(1) Find the coordinates of two other points on the line joining (−1, 4) and (1, 2).

(2) Prove that the points (1, 2), (2, 4), (3, 6) lie on the same line. Find the coordinates of two other points on this line.

(3) $y_1, y_2, y_3, \ldots$ is an arithmetic sequence. Prove that the points with coordinates $(1, y_1), (2, y_2), (3, y_3), \ldots$ all lie on the same line.

(4) $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ are arithmetic sequences. Prove that the points with coordinates $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots$ all lie on the same line.

Slope of a line

We have seen that for any two points on a line not parallel to either axis, the change in y-coordinates is the change in x-coordinate multiplied by a fixed number. This can be said in another form:

For any two points on a line not parallel to either axis, the change in y-coordinates divided by the change in x-coordinates gives the same number

This number is called the **slope** of the line.

For example, let's look at the line joining (2, 8), (4, 9). As we move from the first point to the second

change in x-coordinate is $4 - 2 = 2$

change in y-coordinate is $9 - 8 = 1$

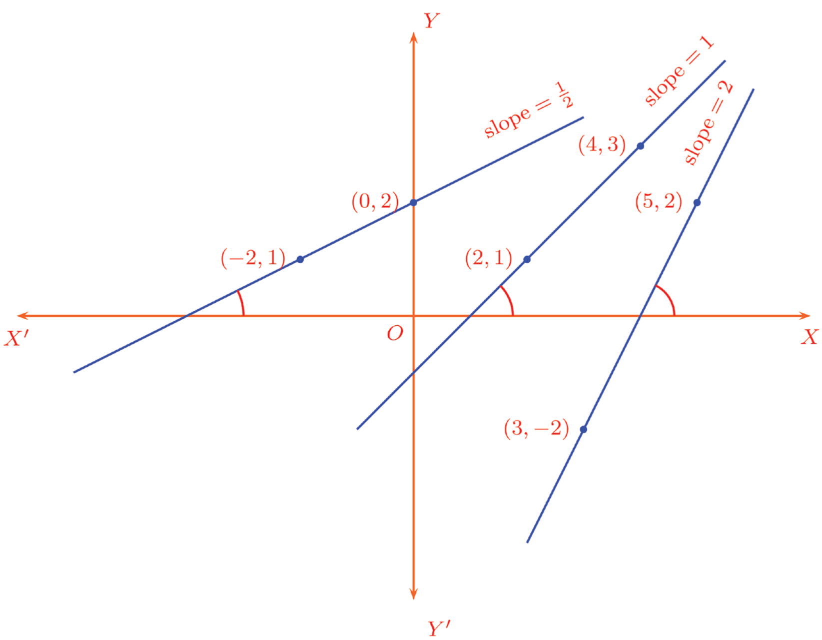
The number got by dividing the change in y-coordinate by the change in x-coordinate is $\frac{1}{2}$. So, the slope of the line joining (2, 8) and (4, 9) is $\frac{1}{2}$.

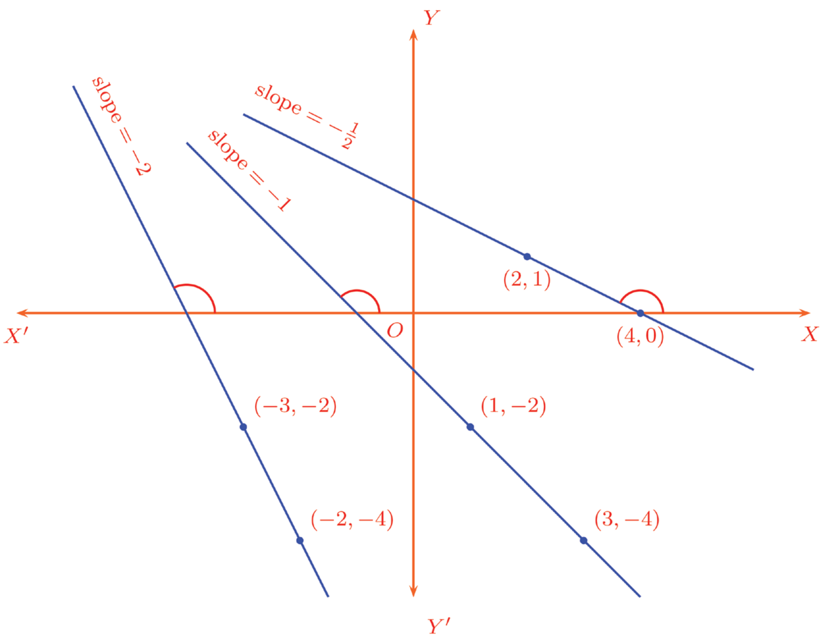
What about the line joining (2, 8) and (4, 6)?

The slope is

$$\frac{6-8}{4-2} = \frac{-2}{2} = -1$$

To see why this number is called the slope, see these pictures:





Do you see that in both pictures as slope increases, the angle on the right between the *x*-axis and the portion of the line above the *x*-axis increases?

In the first picture, the slopes are positive numbers and the angles are less than a right angle; and in the second picture, the slopes are negative and the angles are larger than a right angle.

Thus the number we named slope actually measures the slope of the line with respect to the *x*-axis.

We have seen that if we know the coordinates of two points on a line, we can compute the coordinates of other points on the line; we can also do this if we know the coordinates of one point on the line and the slope of the line instead.

For example, look at this problem:

> The slope of a line passing through (1, 6) is $\frac{3}{4}$. What is the *y*-coordinate of the point on this line with *x*-coordinate 4?

The slope of the line is $\frac{3}{4}$ means, as we move from one point of the line to another, the change in *y*-coordinate is $\frac{3}{4}$ of the change in *x*-coordinate, isn't it?

On this line, as we move from (1, 6) to the point with *x*-coordinate 4, the change in *x*-coordinate is 4 – 1 = 3. So, the change in *y*-coordinate is

$$3 \times \frac{3}{4} = 2\frac{1}{4}$$

So, the y-coordinate is

$$6 + 2\frac{1}{4} = 8\frac{1}{4}$$

Thus $\left(4, 8\frac{1}{4}\right)$ is a point on this line.

We can find other points on the line by taking different numbers as the x-coordinate.

To get the slope of a line drawn in GeoGebra, select Slope and click on the line

(1) Calculate the slope of the line joining each pair of points below:

(i) (2,3), (4, 5) (ii) (2,3), (4, 1) (iii) (1, 1), (−1, −1) (iv) (0, 1), (1, 0)

(2) The slope of a line passing through (2, 5) is $-\frac{2}{3}$. Find the coordinates of two other points on this line.

(3) The slope of a line passing through (3, 1) is $-\frac{1}{2}$. Check whether the points given below are on the line or not:

(i) (5, 2) (ii) (1, 0) (iii) (4, 3) (iv) (2, −1)

(4) We have seen that if $y_1, y_2, y_3, \ldots$ is an arithmetic sequence, then the points $(1, y_1), (2, y_2), (3, y_3), \ldots$ are all on the same line. What is the relation between the slope of this line and the common difference of the arithmetic sequence?

(5) We have seen that if $x_1, x_2, x_3, \ldots$ and $y_1, y_2, y_3, \ldots$ are arithmetic sequences, then the points with coordinates $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots$ all lie on the same line. What is the relation between the slope of this line and the common differences of the arithmetic sequences?

(6) Find another point on the line through (1, 3) with slope $\frac{1}{2}$ and another point on the line through the same point with slope −2. Prove that the lines are perpendicular.

(Hint: Pythagoras Theorem)

Equation of a line

We have seen that if we know the coordinates of one point on a line and the slope of the line, then from the x-coordinate of a point on the line we can calculate its y-coordinate.

For example, see this problem:

The slope of a line through (2, 4) is $\frac{2}{3}$. What is the y-coordinate of the point on this line, with x-coordinate 11?

How do we do this?

(i) As we move from one point of the line to another, the change in y-coordinate is $\frac{2}{3}$ times the change in x-coordinate.

(ii) If we denote the y-coordinate of the point on the line as y with x-coordinate 11, then $(11, y)$ is a point on the line.

(iii) The x-change from (2, 4) to (11, y) is $11 - 2 = 9$ and the y-change is $y - 4$.

(iv) $y - 4 = \frac{2}{3} \times 9 = 6$

(v) $y = 4 + 6 = 10$

So, the y-coordinate of the point with x-coordinate 11, on the line is 10.

In this way, if we know the x-coordinate of a point on this line, we can calculate the y-coordinate.

In other words, there is a relation between the x and y coordinates of any point on this line.

To find this relation, let's take the coordinates of a point on the line as (x, y) and proceed as before:

(i) As we move from one point on the line to another, the change in y-coordinates is $\frac{2}{3}$ times the change in x-coordinates.

(ii) (x, y) is a point on the line.

Physics, algebra and geometry

Suppose a body moves such that the distance travelled is 10 metres in the first second, 15 metres in the next second, 20 metres in the second after and goes on increasing like this. So, its speed also is increasing every second, as 10 m/s during the first second, 15m/s during the next second, 20 m/s during the second after that and so on.

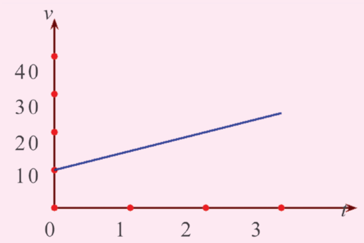
In other words, the speed increases by 5 m/s every second. In the language of physics, the body has an *acceleration* of 5 metres per second.

We can use the algebraic equation

$$v = 10 + 5t$$

to calculate the speed v of this body at time t.

Now let's mark the various t and v along a pair of perpendicular lines and draw the graph of this relation between time and speed:



The slope of this line is 5. Here the slope is the rate at which v changes with respect to t; that is, the acceleration.

(iii) From (2, 4) to (x, y), the x-difference is $x - 2$ and the y-difference is $y - 4$

(iv) $y - 4 = \frac{2}{3}(x - 2)$

Thus we see this:

For any point (x, y) on the line through (2, 4) with slope $\frac{2}{3}$, we have $y - 4 = \frac{2}{3}(x - 2)$.

The relation between the coordinates of every point on the line is called the **equation of the line.**

As we saw above, the equation of the line through (2, 4) with slope $\frac{2}{3}$ is

$$y - 4 = \frac{2}{3}(x - 2)$$

We can rewrite this equation without fractions like this:

$$3(y - 4) = 2(x - 2)$$

That is

$$3y - 12 = 2x - 4$$

This can be simplified as

$$2x - 3y + 8 = 0$$

So we can also say this:

The equation of the line through (2, 4) with slope $\frac{2}{3}$ is

$$2x - 3y + 8 = 0$$

Now look at this problem:

What is the equation of the line joining the points (–1, 2) and (1, –2)?

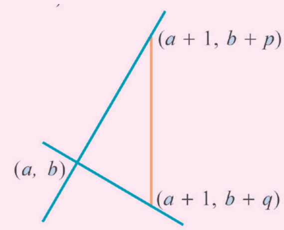
In the line joining (1, 3) and (5, 6) and also in the line joining (2, 5) and (6, 8), the y-difference divided by the x-difference is $\frac{3}{4}$. Draw these lines in GeoGebra. What is the relation between these lines?

Slope and perpendicularity

It is not difficult to see that the slopes of *parallel* lines are equal. What is the relation between the slopes of *perpendicular* lines?

Suppose that lines of slopes p and q are perpendicular to each other. Let's take their point of intersection as (a, b).

Then the point $(a + 1, b + p)$ is on the first line and the point $(a + 1, b + q)$ is on the second line (why?)



Since the lines are perpendicular, the points (a, b), $(a + 1, b + p)$, $(a + 1, b + q)$ are the vertices of a right-angled triangle. The hypotenuse is the line joining the second and third points. So, the squares of the lengths of the perpendicular sides of this triangle are $p^2 + 1$ and $q^2 + 1$ and the length of the hypotenuse is $|p - q|$.

This gives $(p^2 + 1) + (q^2 + 1) = (p - q)^2$

Simplifying, this gives

$$2 = -2pq$$

which means

$$pq = -1$$

Thus, for lines perpendicular to each other, the slope of one is the negative of the reciprocal of the other.

We can think like this:

(i) The slope of the line is

$$\frac{2-(-2)}{-1-1} = \frac{4}{-2} = -2$$

(ii) As we move from one point on the line to another, the change in y-coordinates is the change in x-coordinates multiplied by −2.

(iii) Let's denote the coordinates of a point on the line as (x, y).

(iv) The x-change from (−1, 2) to (x, y) is $x - (-1) = x + 1$ and the y-change is $y - 2$.

(v) $y - 2 = -2(x + 1)$

Thus the equation of the line is

$$y - 2 = -2(x + 1)$$

This we can write as

$$y - 2 = -2x - 2$$

This can be simplified as

$$2x + y = 0$$

In GeoGebra, typing 2x − 3y + 8 = 0 as Input will give the graph of the line whose equation is this. Make three sliders a, b, c and type ax + by + c = 0 as Input. See the changes in the graph as the sliders are moved.

Polynomials and equations

Remember drawing graphs of polynomials in Class 9? How did we draw the graph of the polynomial $p(x) = 2x + 1$?

That method can now be described like this:

Took several numbers as x and calculated $2x + 1$ for each of these.

Marked those points with these pairs of numbers as coordinates and joined them.

Thus the equation of the line drawn is

$$y = 2x + 1$$

In general, the graph of the first degree polynomial

$$p(x) = ax + b$$

is the line whose equation is

$$y = ax + b$$

Draw the line joining (3, 5) and (6, 7) in GeoGebra. The equation of the line can be seen by clicking on View and selecting Algebra

(1) Find the equation of the line joining each pair of points below:

(i) (0, 0), (1, 1) (ii) (0, 0), (1, −1)

(iii) (1, 0), (0, 1) (iv) (−1, 0), (0, −1)

(2) (i) Find the equation of the line joining (−1, 3) and (2, 5).

(ii) Prove that if (x, y) is a point on this line, then so is $(x + 3, y + 2)$.

(3) (i) Find the equation of the line joining $(-1, 1)$ and $(2, 7)$.

(ii) Prove that for any number x, the point $(x, 2x + 3)$ is on this line.

(4) Prove that for any point on the line joining the origin and the point $(1, 2)$, the y-coordinate is twice the x-coordinate.

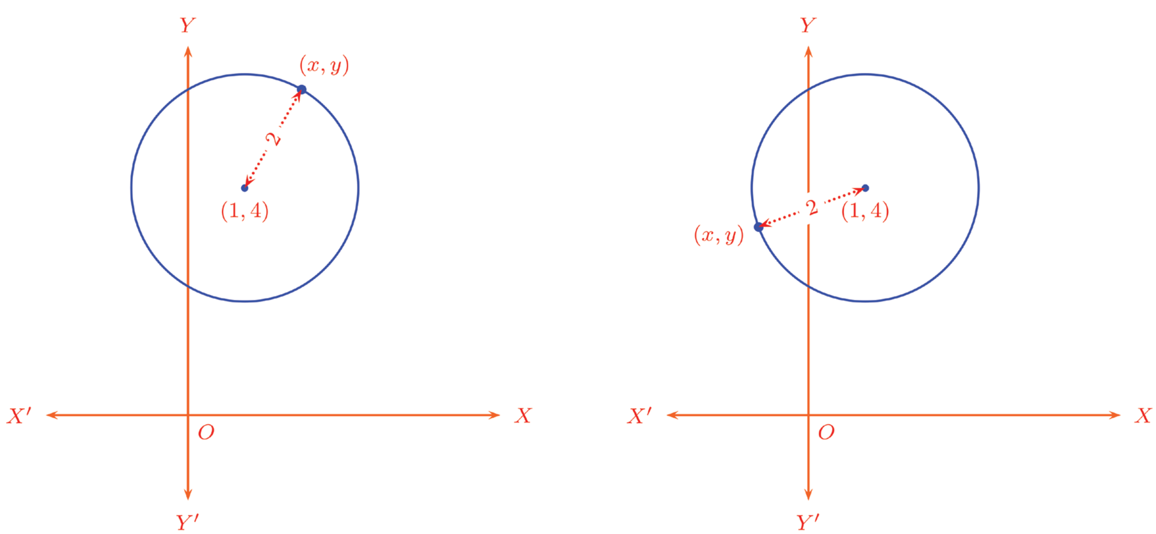
(5) Prove that for any point on the line joining $(2, 0)$ and $(0, 3)$, the sum of half the x-coordinate and one-third the y-coordinate is 1.

Equation of a circle

We can join any two points to draw a line. And if we know the coordinates of these points, we can find the equation of the line also; that is the algebraic form of the relation between the coordinates of any point on the line.

Like this, if we fix a point and a number, we can draw a circle with the point as centre and the number as radius. If we know the coordinates of the centre and the number which is the radius, can we find the relation between the coordinates of every point on the circle?

For example, let's look at the circle with centre $(1, 4)$ and radius 2. For any point on this circle, the distance from the centre is 2:



We have seen that the square of the distance between the points (1, 4) and (x, y) is $(x-1)^2 + (y-4)^2$.

If (x, y) is a point on the circle, then this square is $2^2 = 4$. That is,

$$(x-1)^2 + (y-4)^2 = 4$$

Thus the coordinates of any point on the circle satisfies this equation. On the other hand, if any pair x, y of numbers satisfy this equation, then the point with these as coordinates is at a distance $\sqrt{4} = 2$ from the point (1, 4) and so it is a point on the circle.

Thus this is the equation of the circle with centre (1, 4) and radius 2.

We can also expand this and write

$$x^2 + y^2 - 2x - 8y + 13 = 0$$

So, what is the equation of the circle with centre at the origin and radius 1?

If we take the coordinates of a point on this circle as (x, y), the square of the distance from the origin to this, which is $x^2 + y^2$ must be equal to the square of the radius, which is 1. So, the equation of this circle is

$$x^2 + y^2 = 1$$

(1) Find the equation of the circle with centre at the origin and radius 5. Write the coordinates of eight points on this circle.

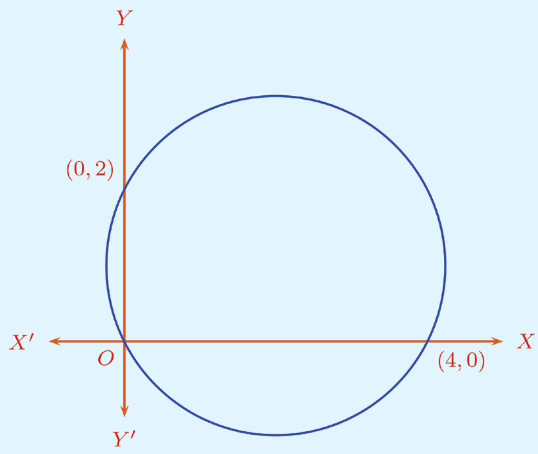
(2) A circle is drawn with the line joining (2, 3) and (4, 7) as diameter.

(i) What are the coordinates of the centre of this circle?

(ii) What is its radius?

(iii) Write the equation of the circle.

(3) What is the equation of the circle in the picture below?

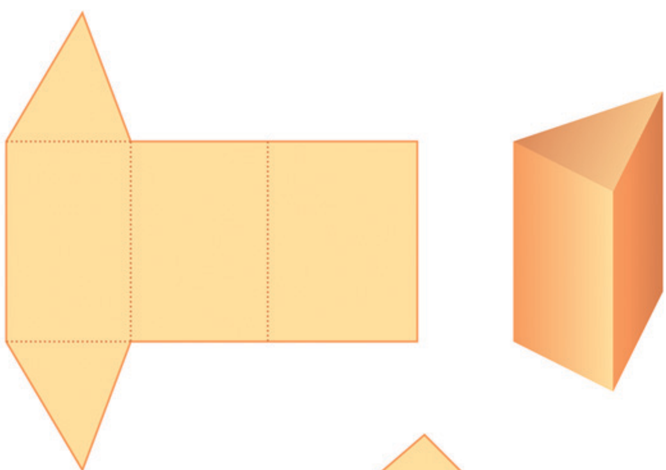


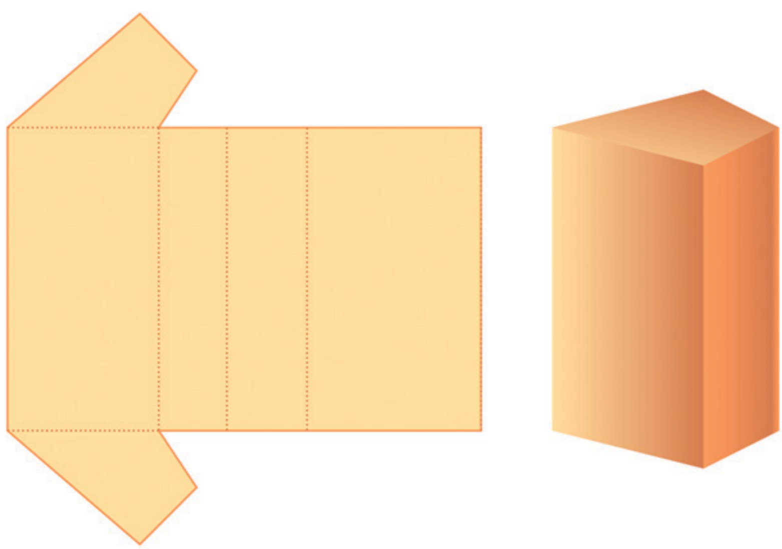
(4) The equation of a circle is $x^2 + y^2 - 2x - 4y - 11 = 0$. Find the coordinates of its centre and the radius.

12 SOLIDS

Pyramids

We can make prisms by cutting thick paper in various ways and pasting the edges.

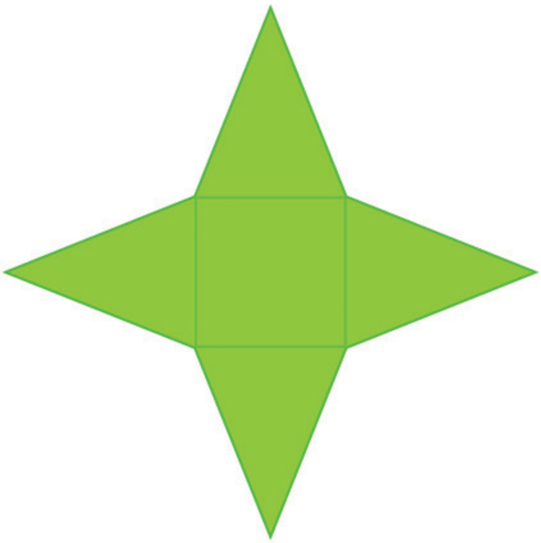




And we have learnt much about them.

Let's make another kind of solid:

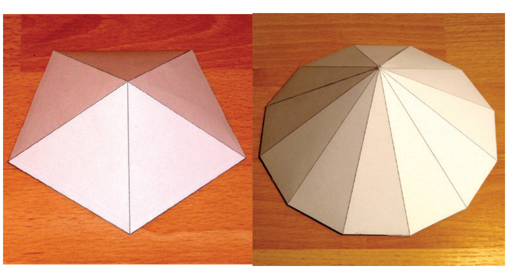
First, cut out a figure like this in paper.



Pyramids in GeoGebra

We have seen in Class 9 how prisms can be drawn in GeoGebra. Now let's see how pyramids can be drawn. Open 3D Graphics and make the necessary preparations (See the section Solids in GeoGebra of the lesson Prisms in the Class 9 textbook). Draw a square in Graphics. Choose Extrude to Pyramid or Cone in 3D Graphics and click on the square. In the window opening up, type the height of the pyramid (we can also make a slider and give its name as the height).

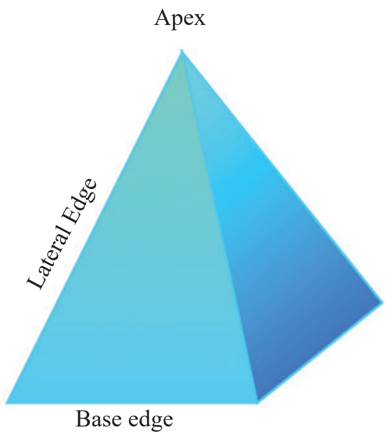
A square in the middle and four triangles around it; all four of them are isosceles triangles and they are equal. Now fold and paste as shown below.



What shape is this? Can't be called a prism; prisms have two equal bases and rectangles on the sides. In the shape we have made now, we have a square at the bottom, a point on top and triangles all around.

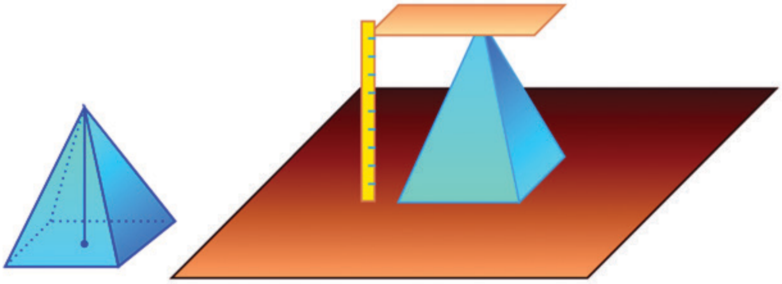
Instead of a square, the base can be some other rectangle, a triangle or some other polygon. Try! (It is better looking when the base is a regular polygon).

Such a solid is generally called a **pyramid**.



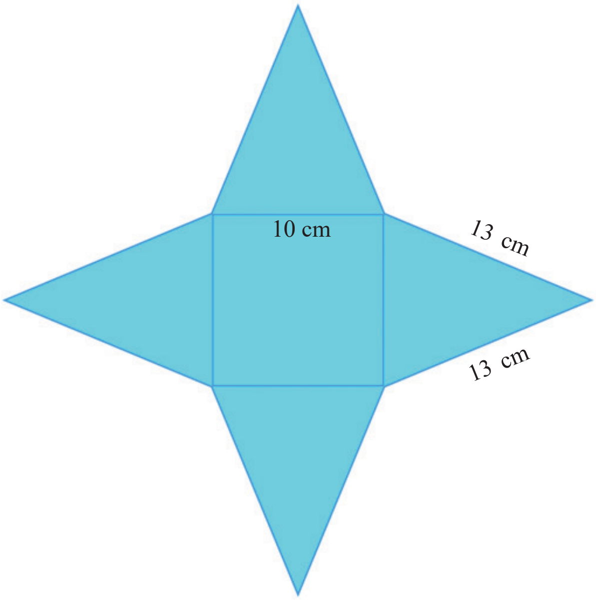
The sides of the polygon forming the base of a pyramid are called **base edges** and the other sides of the triangles are called **lateral edges**. The topmost point of a pyramid is called its **apex**.

The height of a prism is the distance between its bases, isn't it? The height of a pyramid is the perpendicular distance from the apex to the base.



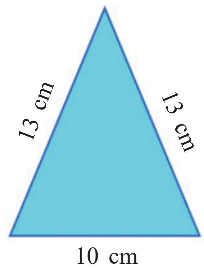
Area

What is the surface area of a square pyramid of base edges 10 centimetres and lateral edges 13 centimetres? The surface area is the area of paper needed to make it. How will it look if we cut this pyramid open and lay it flat?

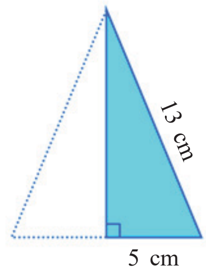


Let's see how GeoGebra helps us to see the 'cut and spread' shape of a pyramid. Make a pyramid in 3D graphics as described earlier. Choose Net and click on the pyramid. We get the shape of the paper used to make it (it is called the net of the solid). We also get a slider in Graphics. By moving the slider, we can see how the pyramid is made from the net. We can also hide the original pyramid by clicking on the pyramid in the Algebra window.

The area of the square is easily seen to be 100 square centimetres. What about the triangles?



The area of the triangle is half the product of its base and height, isn't it?



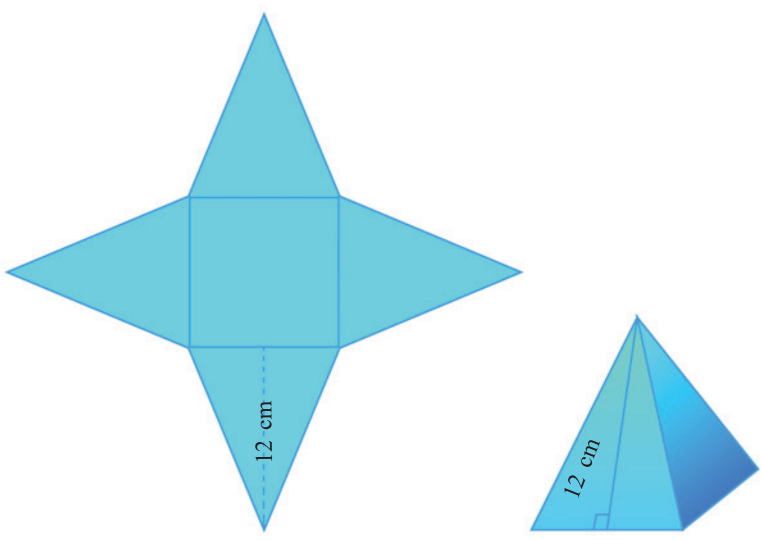
For that, we need the height of the triangle. Since the triangle is isosceles, this altitude bisects the base.

So using Pythagoras Theorem, the height of the triangle is

$$\sqrt{13^2 - 5^2} = 12 \text{ centimetres}$$

Thus the area of the triangle is $5 \times 12 = 60$ square centimetres.

What will be the height of the triangle when the paper is turned into a pyramid?

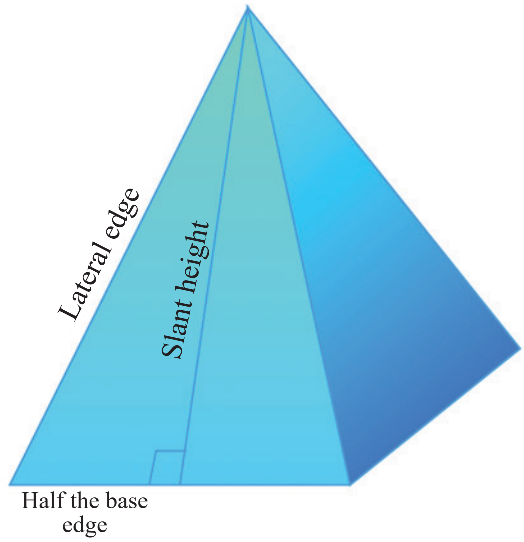


Height and slant height

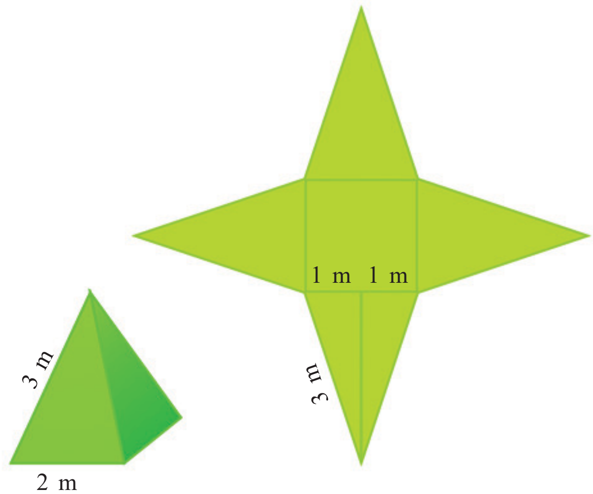
Draw a pyramid in GeoGebra. Click Midpoint or Centre to mark the midpoints of a base edge and the midpoint of a diagonal of the base. Use Segment to mark the height and slant height of the pyramid. Use Polygon to make the right triangle with height, slant height and half the base edge as sides. Using Net, the pyramid can be cut and spread. We can also hide the pyramid.

This length is called the **slant height** of the pyramid.

We have seen the relation between the base edge, lateral edge and slant height of a pyramid in the problem we did just now. As shown in the picture on the right, there is a right triangle on each side of the pyramid - its perpendicular sides are the slant height and half the base edge, the hypotenuse is a lateral edge.



Now do this problem: what is the surface area of a square pyramid with base edges 2 metres and lateral edges 3 metres?



The base area is 4 square metres. To compute the areas of lateral faces, we need the slant height. In the right triangle mentioned above, one side is half the base edge, that is, 1 metre and the hypotenuse is the lateral edge of 3 metres.

So, the slant height is

$$\sqrt{3^2 - 1^2} = 2\sqrt{2} \text{ metres}$$

Using this, the area of each triangular face is

$$\frac{1}{2} \times 2 \times 2\sqrt{2} = 2\sqrt{2} \text{ square metres.}$$

So, the surface area of the pyramid is

$$4 + (4 \times 2\sqrt{2}) = 4 + 8\sqrt{2} \text{ square metres.}$$

If not satisfied with this, a calculator can be used, (or an approximate value of $\sqrt{2}$ recalled) to compute this as 15.31 square metres.

(1) A square of side 5 centimetres, and four isosceles triangles of base 5 centimetres and height 8 centimetres, are to be put together to make a square pyramid. How many square centimetres of paper are needed?

(2) A toy is in the shape of a square pyramid of base edge 16 centimetres and slant height 10 centimetres. What is the total cost of painting 500 such toys, at 80 rupees per square metre?

(3) The lateral faces of a square pyramid are equilateral triangles and the length of a base edge is 30 centimetres. What is its surface area?

(4) The perimeter of the base of a square pyramid is 40 centimetres and the total length of all its edges is 92 centimetres. Calculate its surface area.

(5) Can we make a square pyramid with the lateral surface area equal to the base area?

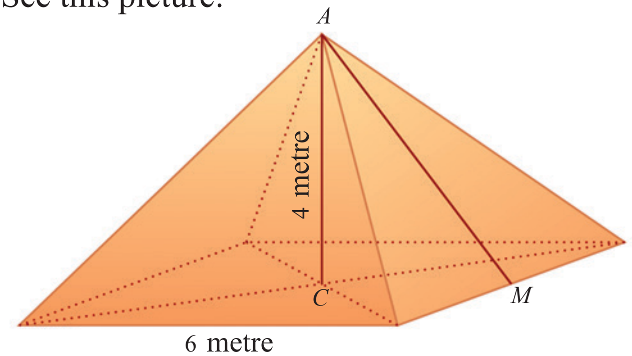
Height and slant height

The height of a pyramid is often an important measure. See this problem:

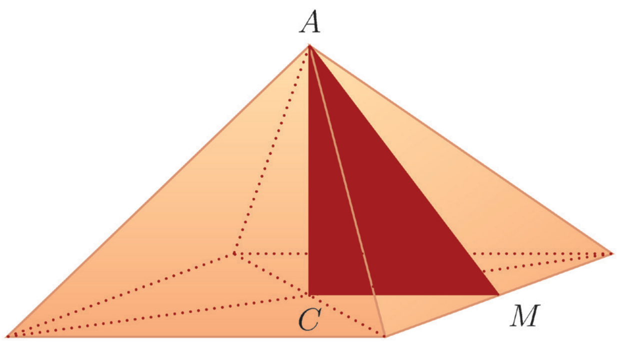
A tent is to be made in the shape of a square pyramid of base edges 6 metres and height 4 metres. How many square metres of canvas is needed to make it?

To calculate the area of the triangular faces of the tent, we need the slant height. How do we compute it using the given specifications?

See this picture:



The slant height we need is AM. Joining CM, we get a right triangle with AM as hypotenuse. What is the length of CM in it?



Pyramids of Egypt

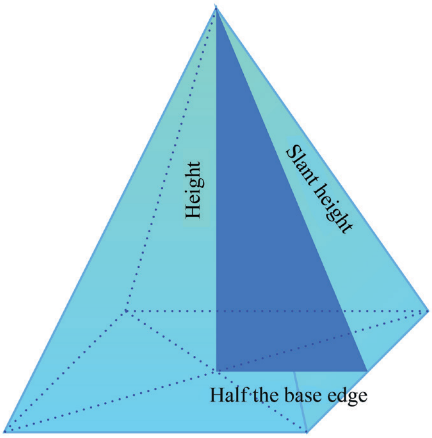
The very word pyramid brings to our mind the great pyramids of Egypt. 138 such pyramids are found in various parts of Egypt. Many of them were built around 2000 BCE.

From the picture, $AM = \sqrt{3^2 + 4^2} = 5$ metres.

So, to make the tent, four isosceles triangles of base 6 metres and height 5 metres are needed. Their total area is $4 \times \frac{1}{2} \times 6 \times 5 = 60$ square metres.

So this much canvas is needed to make the tent.

In this problem, we have found something which is true in the case of all square pyramids. Within every square pyramid, we can imagine a right triangle with perpendicular sides as the height of the pyramid and half the base edge and the hypotenuse as the slant height.

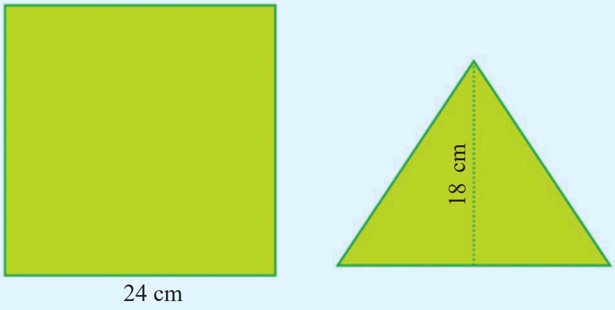


Great Pyramid

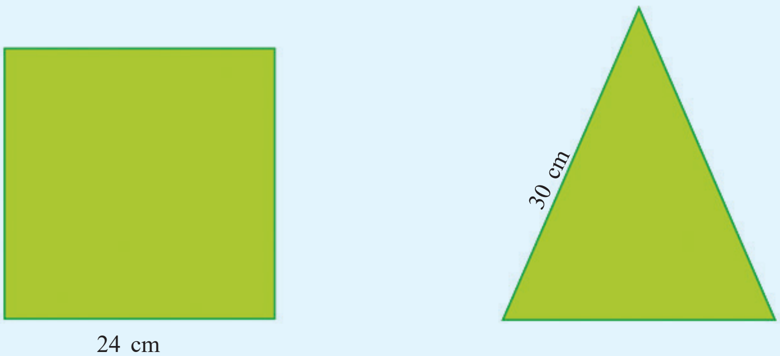
The largest pyramid in Egypt is the Great Pyramid of Giza.

Its base is a square of almost half a lakh square metres and its height is about 140 metres. It is estimated that about 20 years would have been needed to complete it. These royal tombs built with huge blocks of stones stacked with precision to end in a point are living symbols of human labour, engineering skill and mathematical knowledge.

(1) Using a square and four triangles with dimensions as specified in the picture, a pyramid is made.
What is the height of this pyramid?



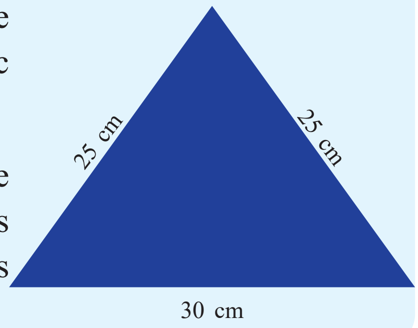
What if the square and triangles are like this?



(2) A square pyramid of base edge 10 centimetres and height 12 centimetres is to be made of paper. What should be the dimensions of the triangles?

(3) Prove that in any square pyramid, the squares of the height, slant height and lateral edge are in arithmetic sequence.

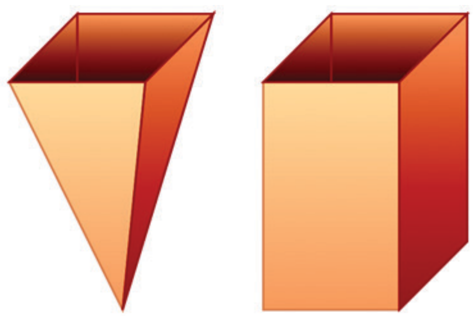
(4) A square pyramid is to be made with the triangle shown here as a lateral face. What would be its height? What if the base edge is 40 centimetres instead of 30 centimetres?



Can we make a square pyramid with any four equal isosceles triangles?

Volume of a pyramid

We have seen that the volume of any prism is equal to the product of the base area and the height. What about the volume of a pyramid?



Let's take the case of a square pyramid. Make a hollow square pyramid with thick paper and also a square prism of the same base and height.

Fill the pyramid with sand and transfer it to the prism. Measure the height of the sand in the prism and see what fraction of the height of the prism it is. A third, isn't it? So to fill the prism, how many times should we fill the pyramid?

Thus we see that the volume of the prism is three times the volume of the pyramid. (A mathematical explanation of this is given at the end of this lesson).

We have seen in Class 9 that the volume of a prism is equal to the product of the base area and the height.

So what can we say about the volume of a square pyramid?

Pyramid Volume

Draw a square pyramid and a square prism of the same base and with the same height in GeoGebra. To distinguish between them, change the colour of the pyramid and make Opacity 100. (Object properties → Colour). Find their volumes using Volume. What is the relation between them? Change the base and height and see.

The volume of a square pyramid is equal to a third of the product of the base area and the height

For example, the volume of a square pyramid of base edge 10 centimetres and height 8 centimetres is $\frac{1}{3} \times 10^2 \times 8 = 266\frac{2}{3}$ cubic centimetres.

A metal cube of edge 15 centimetres is melted and recast into a square pyramid of base edge 25 centimetres. What is its height?

The volume of the cube is 15^3 cubic centimetres.

The volume of the square pyramid is also this. And the volume of a pyramid is a third of the product of the base area and height.

Since the base area of our pyramid is 25^2 square centimetres, a third of the height is $\frac{15^3}{25^2}$ and so the height is,

$$3 \times \frac{15^3}{25^2} = 16.2 \text{ centimetres.}$$

(1) What is the volume of a square pyramid of base edge 10 centimetres and slant height 15 centimetres?

(2) Two square pyramids have the same volume. The base edge of one is half that of the other. How many times is the height of the second pyramid the height of the first?

(3) The base edges of two square pyramids are in the ratio 1 : 2 and their heights in the ratio 1 : 3. The volume of the first is 180 cubic centimetres. What is the volume of the second?

(4) All edges of a square pyramid are 18 centimetres. What is its volume?

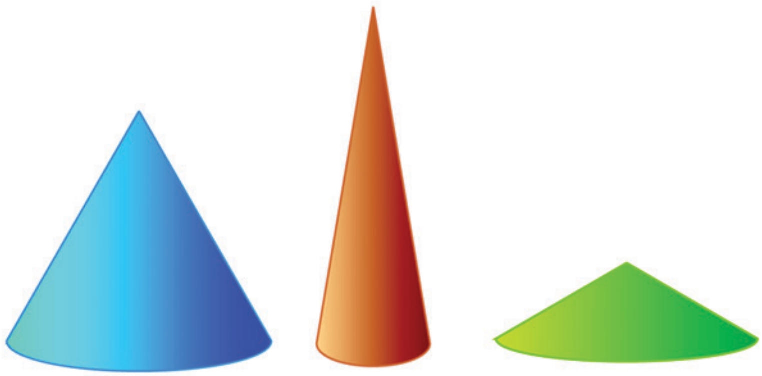
(5) The slant height of a square pyramid is 25 centimetres and its surface area is 896 square centimetres. What is its volume?

(6) All edges of a square pyramid are of the same length and its height is 12 centimetres. What is its volume?

(7) What is the surface area of a square pyramid of base perimeter 64 centimetres and volume 1280 cubic centimetres?

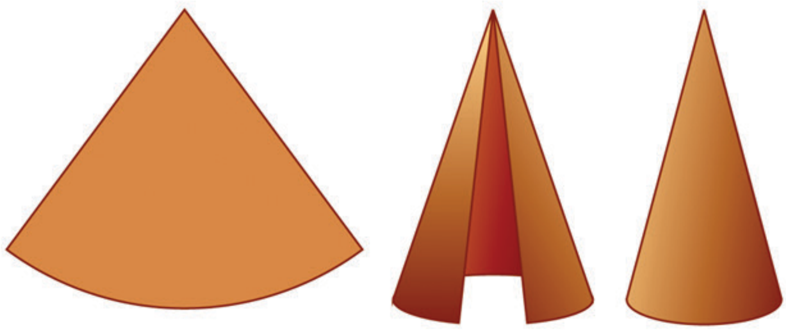
Cone

Cylinders are prism-like solids with circular bases. Similarly, we have pyramid-like solids with circular bases:



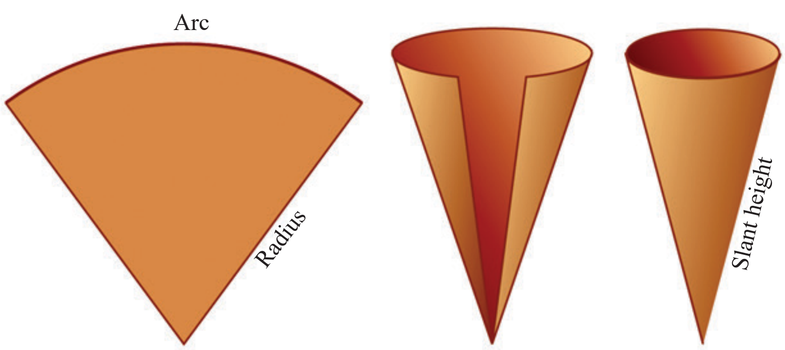
They are called **cones**.

We can make a cylinder by rolling up a rectangle. Likewise, we can make a cone by rolling up a sector of a circle.



What is the relation between the dimensions of the sector we start with and the cone we end up with?

The radius of the sector becomes the slant height of the cone. The arc length of the sector becomes the circumference of the base of the cone.



Cone

We can draw cones in GeoGebra, just as we drew pyramids. Draw a circle in Graphics and in 3D Graphics, use Extrude to Pyramid or Cone. The base radius and height can be changed using sliders.

We often specify the size of a sector in terms of the central angle. See this problem:

From a circle of radius 12 centimetres, a sector of central angle 45° is cut out and made into a cone. What are the slant height and base radius of this cone?

The slant height of the cone is the radius of the circle itself: 12 centimetres. What about its base radius?

45° is $\frac{1}{8}$ of 360°. And the arc length of a sector is proportional to the central angle. So this arc length is $\frac{1}{8}$ of the circumference of the full circle.

This arc becomes the base circle of the cone. Thus the circumference of the base circle of the cone is $\frac{1}{8}$ of the circumference of the larger circle from which the sector was cut out. Since radii of circles are proportional to their circumferences, the radius of the smaller circle is $\frac{1}{8}$ of the radius of the large circle. Thus the radius of the base of the cone is

$$\frac{1}{8} \times 12 = 1.5 \text{ centimetres.}$$

How about a question in the reverse direction?

How do we make a cone of base radius 5 centimetres and slant height 15 centimetres?

To make a cone, we need a sector. Since the slant height is to be 15 centimetres, the sector must be cut out from a circle of radius 15 centimetres.

What should be its central angle?

The radius of the small circle forming the base of the cone is $\frac{5}{15} = \frac{1}{3}$ of the radius of the large circle from which the sector is to be cut out. (How do we get this?) So, the circumference of the small circle is also $\frac{1}{3}$ of the circumference of the large circle.

The circumference of the small circle is the arc length of the sector. Thus the arc of the sector is $\frac{1}{3}$ of the circle from which it is cut out. So its central angle must be $360° \times \frac{1}{3} = 120°$.

(1) What are the radius of the base and slant height of a cone made by rolling up a sector of central angle 60° cut out from a circle of radius 10 centimetres?

(2) What is the central angle of the sector to be used to make a cone of base radius 10 centimetres and slant height 25 centimetres?

(3) What is the ratio of the base-radius and slant height of a cone made by rolling up a semicircle?

Curved surface area

As in the case of a cylinder, a cone also has a curved surface - the part which rises up at a slant. The area of this curved surface is the area of the sector used to make the cone. (For a cylinder also, the area of the curved surface is the area of the rectangle rolled up to make it, isn't it?)

For a cone drawn in GeoGebra, the curved surface area can be seen under Surface in the Algebra window.

See this problem:

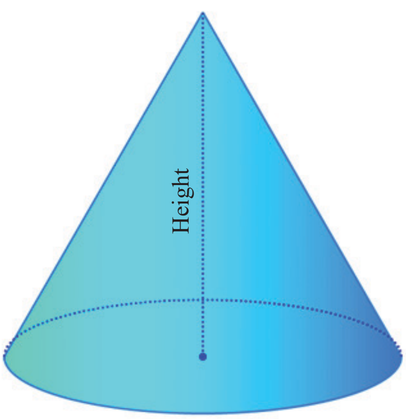
To make a conical hat of base radius 8 centimetres and slant height 30 centimetres, how much square centimetres of paper do we need?

What we need here is the area of the sector we roll up to make this hat. Since the slant height is to be 30 centimetres, we must cut out the sector from a circle of this radius. Also the radius of the small circle forming the base of the cone must be 8 centimetres, that is $\frac{8}{30} = \frac{4}{15}$ of the radius of the large circle from which the sector is cut out. So the circumference of the small circle is also the same fraction of the circumference of the large circle. The arc length of the sector is the circumference of the small circle. Thus the sector to be cut out is $\frac{4}{15}$ of the full circle. So, its area is that fraction of the area of the circle; that is,

$$\pi \times 30^2 \times \frac{4}{15} = \pi \times 2 \times 30 \times 4 = 240\pi \text{ square centimetres.}$$

Thus we need 240π square centimetres of paper to make the hat (it can be computed as approximately 754 square centimetres).

As in a pyramid, the height of a cone is the perpendicular distance from the apex to the base, and it is the distance between the apex and the centre of the base circle.



Curved surface

The area of the curved surface of a cone is the area of the sector used to make it. If we take the base radius of the cone as r and its slant height as l, then the radius of the sector is l and its central angle is $\frac{r}{l} \times 360°$. So its area is

$$\frac{1}{360} \times \left(\frac{r}{l} \times 360\right) \times \pi l^2 = \pi r l$$

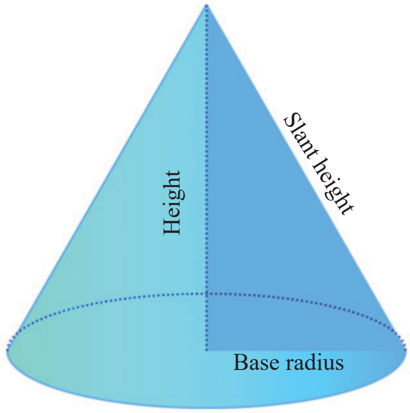
(Recall the computation of the area of a sector in Class 9).

Thus the area of the curved surface of a cone is half the product of the base circumference and the slant height.

Again, as in the case of a square pyramid, the height is related to the slant height via a right triangle.

For example, in a cone of base radius 5 centimetres and height 10 centimetres the slant height is

$\sqrt{5^2 + 10^2} = \sqrt{125} = 5\sqrt{5}$ centimetres.



(1) What is the area of the curved surface of a cone of base radius 12 centimetres and slant height 25 centimetres?

(2) What is the surface area of a cone of base diameter 30 centimetres and height 40 centimetres?

(3) A conical fire work is of base diameter 10 centimetres and height 12 centimetres. 10000 such fireworks are to be wrapped in colour paper. The price of the colour paper is 2 rupees per square metre. What is the total cost?

(4) Prove that for a cone made by rolling up a semicircle, the area of the curved surface is twice the base area.

Volume of a cone

To find the volume of a cone, we can do an experiment similar to the one we did to find the volume of a square pyramid. Make a cone and a cylinder of the same base and height. Fill the cone with sand and transfer it to the cylinder. Here also, we can see that the volume of the cone is a third of the volume of the cylinder. Thus we have the following:

As in the case of square pyramids, draw a cylinder and a cone of the same base and height in GeoGebra. Compare their volumes.

The volume of a cone is equal to a third of the product of the base area and height

(A mathematical explanation of this also is given at the end of this lesson).

For example, the volume of a cone of base radius 4 centimetres and height 6 centimetres is

$$\frac{1}{3} \times \pi \times 4^2 \times 6 = 32\pi \text{ cubic centimetres.}$$

(1) The base radius and height of a cylindrical block of wood are 15 centimetres and 40 centimetres. What is the volume of the largest cone that can be carved out of this?

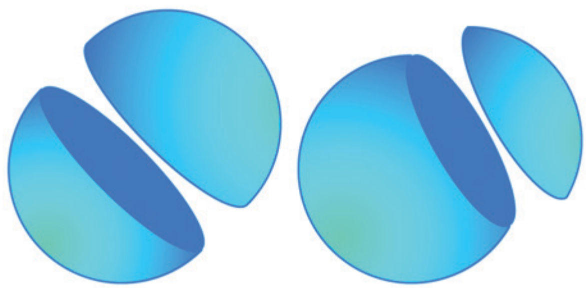
(2) The base radius and height of a solid metal cylinder are 12 centimetres and 20 centimetres. By melting it and recasting, how many cones of base radius 4 centimetres and height 5 centimetres can be made?

(3) A sector of central angle 216° is cut out from a circle of radius 25 centimetres and is rolled up into a cone. What are the base radius and height of the cone? What is its volume?

(4) The base radii of two cones are in the ratio 3 : 5 and their heights are in the ratio 2 : 3. What is the ratio of their volumes?

(5) Two cones have the same volume and their base radii are in the ratio 4 : 5. What is the ratio of their heights?

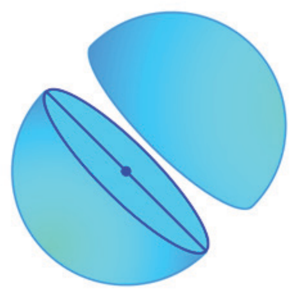
Sphere



Round solids enter our lives in various ways - as the thrill of ball games and as the sweetness of laddus. Now let's look at the mathematics of such solids called **spheres**.

If we slice a cylinder or cone parallel to a base, we get a circle. In whatever way we slice a sphere, we get a circle.

The distance of any point on a circle from the centre is the same. A sphere also has a *centre*, from which the distance to any point on its surface is the same. This distance is called the *radius* of the sphere and double this is called the *diameter*.



If we slice a sphere into exact halves, we get a circle whose centre, radius and diameter are those of the sphere itself.

We cannot cut open a sphere and spread it flat, as we did with other solids. The fact is that we cannot make the surface of a sphere flat without some folding or stretching.

But we can prove that the surface area of a sphere of radius r is $4\pi r^2$ (an explanation is given at the end of the lesson).

We can state this in a different way.

The surface area of a sphere is equal to the square of its radius multiplied by 4π

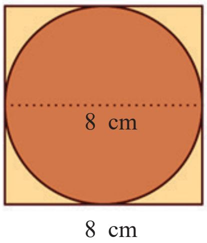
Also, we can prove that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$ (An explanation of this also is given at the end of the lesson).

See this problem:

What is the surface area of the largest sphere that can be carved from a cube of edge 8 centimetres?

We can see from the picture that the diameter of the sphere is the length of an edge of the cube. So, the surface area of the sphere is

$$4\pi \times 4^2 = 64\pi \text{ square centimetres}$$

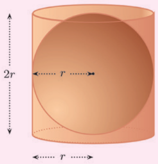


Another problem:

Sphere and cylinder

Consider a cylinder which can cover a sphere precisely. Its base radius is the radius of the sphere and its height is double its radius.

So if we take the radius of the sphere as r, the base radius and height of the cylinder are r and $2r$. So its surface area is



$$(2\pi r \times 2r)+(2\times\pi r^2)=6\pi r^2$$

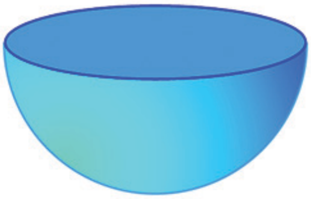
The surface area of the sphere is $4\pi r^2$. Thus the ratio of these surface areas is 3 : 2.

Again, the volume of the cylinder is

$$\pi r^2 \times 2r = 2\pi r^3$$

and the volume of the sphere is $\frac{4}{3}\pi r^3$, so that the ratio of the volumes is also 3 : 2.

A solid sphere of radius 12 centimetres is cut into two equal halves. What is the surface area of each hemisphere?



The surface of the hemisphere consists of half the surface of the sphere and a circle.

Since the radius of the sphere is 12 centimetres, its area is

$$4\pi \times 12^2 = 576\pi \text{ square centimetres}$$

The surface area of the hemisphere is half of this added to the area of the circle.

Since the radius of the circle is 12 centimetres, its area is

$$\pi \times 12^2 = 144\pi \text{ square centimetres}$$

So the surface area of the hemisphere is

$$\frac{1}{2} \times 576\pi + 144\pi = 432\pi \text{ square centimetres}$$

(1) The surface area of a solid sphere is 120 square centimetres. If it is cut into two halves, what would be the surface area of each hemisphere?

(2) The volumes of two spheres are in the ratio 27 : 64. What is the ratio of their radii? And the ratio of their surface areas?

(3) The base radius and length of a metal cylinder are 4 centimetres and 10 centimetres. If it is melted and recast into spheres of radius 2 centimetres each, how many spheres can be made?

(4) A metal sphere of radius 12 centimetres is melted and recast into 27 small spheres of equal size. What is the radius of each small sphere?

(5) From a solid sphere of radius 10 centimetres, the largest cone of height 16 centimetres is carved out. What fraction of the volume of the sphere is the volume of the cone?

(6) A solid sphere is cut into two hemispheres. From one, a square pyramid and from the other a cone, each of maximum possible size are carved out. What is the ratio of their volumes?

(1) What is the speciality of the lateral faces of a square pyramid of maximum volume that can be cut out from a solid hemisphere?

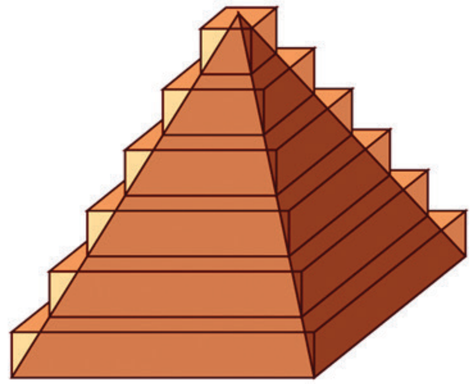
(2) Find the relationship between the volumes of a cylinder, a cone, a square pyramid and a sphere of maximum size that can be carved out of a cube.

Appendix

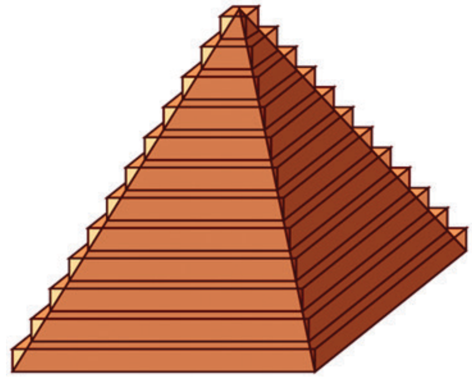
We have seen only the techniques of calculating volumes of pyramids and cones, and also the surface area and volume of a sphere. For those who may be interested in knowing how they are actually got, we give some explanations below.

Volume of a pyramid

We can think of a stack of square plates, of decreasing size as an approximation to a square pyramid.



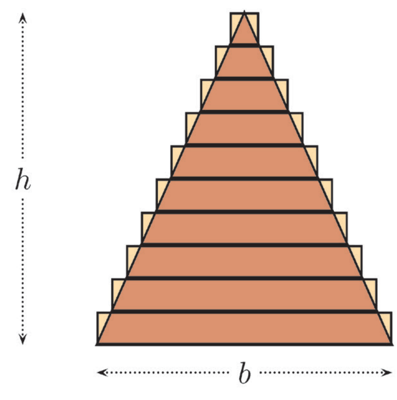
As we decrease the thickness of the plates and increase their number, we get better approximations.



And the sum of the volumes of these plates gets nearer to the volume of the pyramid.

Suppose we use 10 plates, to start with. Each plate is a square prism of small height. Let's use plates of the same height. So, if we take the height of the pyramid as h, each plate is of height $\frac{1}{10}h$. How do we compute the base of each plate?

If we imagine the pyramid and the stack of plates sliced vertically down from the vertex, we get a picture like this.



Starting from the top, we have isosceles triangles of increasing size. Their heights increase at the rate of $\frac{1}{10}h$ for each plate.

Since these triangles are all similar (why?) their bases also increase at the same rate. So, if we take the base edge of the bases of the pyramid to be b, the bases of the triangles starting from the top are $\frac{1}{10}b, \frac{2}{10}b, \ldots, b$.

So, the volumes of the plates are

$$\left(\frac{1}{10}b\right)^2 \times \frac{1}{10}h, \left(\frac{2}{10}b\right)^2 \times \frac{1}{10}h, \ldots, b^2 \times \frac{1}{10}h$$

And their sum?

$$\frac{1}{10}b^2h\left[\frac{1}{10^2} + \frac{2^2}{10^2} + \ldots + \frac{9^2}{10^2} + \frac{10^2}{10^2}\right] = \frac{1}{1000}b^2h(1^2 + 2^2 + 3^2 + \ldots + 10^2)$$

We have seen how such sums can be computed in the section, **Sum of Squares** of the lesson **Arithmetic Sequences**.

$$1^2 + 2^2 + 3^2 + \ldots + 10^2 = \frac{1}{6} \times 10 \times (10 + 1) \times (2 \times 10 + 1)$$

Thus the sum of the volumes

$$\frac{1}{1000}b^2h \times \frac{1}{6} \times 10 \times 11 \times 21 = \frac{1}{6}b^2h \times \frac{10}{10} \times \frac{11}{10} \times \frac{21}{10} = \frac{1}{6}b^2h \times 1.1 \times 2.1$$

Now imagine 100 such plates (we cannot draw it anyway).

The thickness of a plate becomes $\frac{1}{100}h$ and the base edges would be $\frac{1}{100}b, \frac{2}{100}b, \frac{3}{100}b, \ldots, b$. So the sum of the volumes would be

$$\begin{aligned}\frac{1}{100^3}b^2h \times (1^2 + 2^2 + 3^2 + \ldots + 100^2) &= \frac{1}{100^3}b^2h \times \frac{1}{6} \times 100 \times 101 \times 201 \\ &= \frac{1}{6}b^2h \times \frac{100}{100} \times \frac{101}{100} \times \frac{201}{100} \\ &= \frac{1}{6}b^2h \times 1.01 \times 2.01\end{aligned}$$

What if we increase the number of plates to 1000? Without going through detailed computations, we can see that the sum of volumes would be

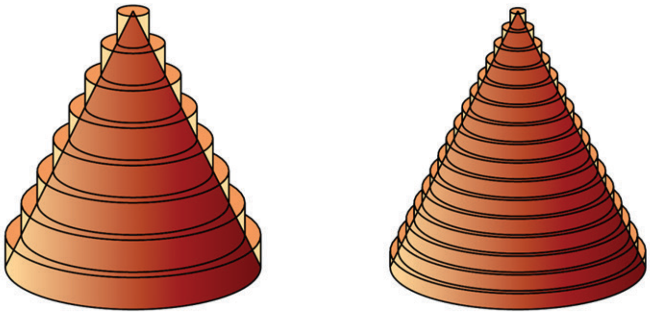
$$\frac{1}{6}b^2h \times 1.001 \times 2.001$$

What is the number to which these sums get closer and closer to?

It is the volume of the pyramid; and it is

$$\frac{1}{6}b^2h \times 1 \times 2 = \frac{1}{3}b^2h$$

Volume of a cone

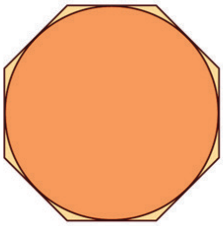


Just as we stacked square plates to approximate a pyramid, we can stack circular plates to approximate a cone.

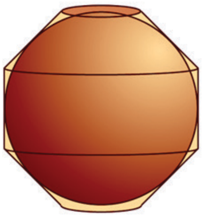
And in much the same way, we can compute the volume of a cone also. (Try!)

Surface area of a sphere

First consider a circle through the middle of the sphere and a regular polygon with its sides touching it;



Now if this figure revolves, we get the sphere and a solid just covering it.

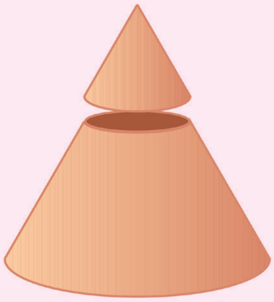


In the picture above, this solid can be split into two frustums and a cylinder.

As we increase the number of sides of the polygon, the covering solid approximates the sphere better.

Small and large

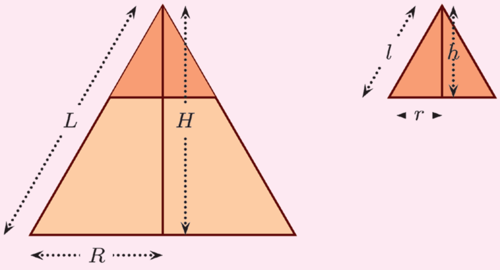
Cutting a cone parallel to the base, we get a small cone on top.



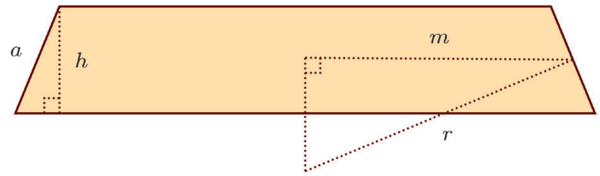
What is the relation between the dimensions of the small and large cones?

If we take the base radius, height and slant height of the large cone as R, H, L and those of the small cone as r, h, l we get

$$\frac{r}{R} = \frac{h}{H} = \frac{l}{L}$$



To compute the area of the curved surface of these frustums, let's consider one of these. Let's take its height as h and the radius of its middle circle as m. Let's also take the radius of the sphere as r and the length of a side of the covering polygon as a. We then have a figure like this.



The two right triangles in the figure are similar and so

$$\frac{m}{r} = \frac{h}{a}$$

which can be written as

$$am = rh$$

The area of the curved surface of the frustum got by revolving this is $2\pi ma$, as shown in the sidebar. *Frustum* and *cylinder*, on the last page. And this is equal to $2\pi rh$, by the above equation; that is, the area of the curved surface of a cylinder of base radius r and height h.

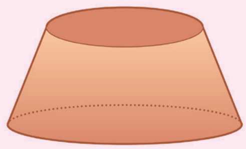
So what do we get? In the solid which approximates the sphere, the curved surface area of each frustum is equal to that of a cylinder of the same height with base radius equal to that of the sphere.

So the curved surface area of the whole approximating solid is equal to the sum of the curved surface areas of all these cylinders. And what do we get on putting together all these cylinders? A large cylinder, just covering the sphere.

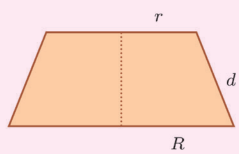
As we increase the number of sides of the polygon covering the circle, it becomes more circle-like; and the solid covering the

Frustum of a cone

If we cut off a small cone from the top of a cone, the remaining piece is called the frustum of a cone.



How do we find the area of the curved surface of a frustum in terms of its slant height and base radii?



Taking the slant heights of the large and small cones as L and l, we get d in the figure above as $d = L - l$. So the surface area of the frustum is,

$$\begin{aligned}\pi RL - \pi rl &= \pi(RL - rl)\\ &= \pi(R(l + d) - rl)\\ &= \pi(Rl + Rd - rl)\end{aligned}$$

Now, as seen earlier, we have $\frac{r}{R} = \frac{l}{L}$.

So that $Rl = rL$. Using this, the area of the curved surface is,

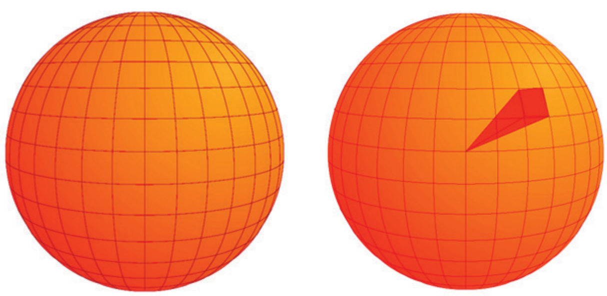
$$\begin{aligned}\pi(rL + Rd - rl) &= \pi(r(L - l) + Rd)\\ &= \pi(rd + Rd)\\ &= \pi(r + R)d\end{aligned}$$

sphere becomes more sphere-like. As seen just now, the curved surface area of any such solid is equal to the curved surface area of a cylinder just covering the sphere. So the surface area of the sphere is also equal to the area of the curved surface of this cylinder. Since the base radius of the cylinder is r and its height is $2r$, the area of its curved surface is

$$2\pi \times r \times 2r = 4\pi r^2$$

Volume of a sphere

See these pictures:

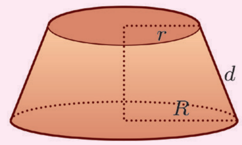


A sphere is divided into cells by horizontal and vertical circles. If we join the corners of such a cell to the centre of the sphere, we get a pyramid-like solid.

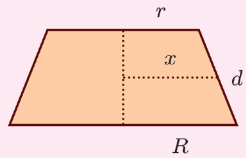
The sphere is made up of such solids joined together; and so the volume of the sphere is the sum of the volumes of these solids. Now if we change each cell into an actual square which touches the sphere, we get a solid which just covers the sphere; and the solid is made up of actual square pyramids.

Frustum and cylinder

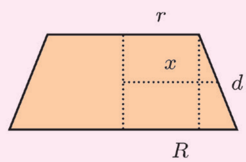
We have seen that the area of the curved surface of a frustum is $\pi(r + R)d$



Taking the radius of the circle round its middle as x, we get a figure like this:



Let's draw one more line:



From the two similar right triangles on the right,

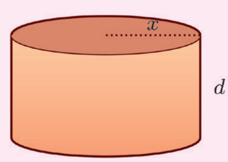
$$\frac{x - r}{R - r} = \frac{1}{2}$$

Simplifying, this gives

$$x = \frac{1}{2}(R + r)$$

So, the area of the curved surface of the frustum can be written $2\pi xd$.

But this is the area of the curved surface of a cylinder of base radius x and height d.



The heights of all these pyramids are equal to the radius of the sphere. If we take it as r and the base area of a pyramid as a, the volume of a pyramid is $\frac{1}{3}ar$. The volume of the solid covering the sphere is the sum of the volumes of all such pyramids. The bases of all such pyramids make up the surface of the solid, so that the sum of the base areas of the pyramids is the surface area of this solid. If it is taken as s, the volume of the solid would be $\frac{1}{3}sr$.

As we decrease the size of the cells and increase their number, the solid covering the sphere approximates the sphere better; and the surface area of the solid gets closer to the surface area of the sphere. Since the surface area of the sphere is $4\pi r^2$, the volume of the solid gets closer to the number.

$$\frac{1}{3} \times 4\pi r^2 \times r = \frac{4}{3}\pi r^3$$

And this is the volume of the sphere.

13 Statistics

Not a correct average

The monthly income of 10 households in a neighbourhood are these:

16500	21700	18600	21050	19500
17000	21000	18000	22000	17500

What is the mean monthly income?

Adding all these and dividing by 10, we get the mean monthly income as 19285 rupees.

Now, if instead of taking all these incomes separately, we had only the mean, then also we can make some conclusions about the general economic status of the households:

- The monthly income of all these households are around 19285 rupees.
- None of the households has a monthly income very much greater or very much less than 19285 rupees.
- The number of households with monthly income greater than 19285 rupees is more or less equal to the number of households with monthly income less than 19285 rupees.

Now suppose someone with a monthly income of 175000 rupees comes to live in the neighbourhood. What is the mean monthly income of the 11 households?

$$\frac{(19285 \times 10) + 175000}{11} \approx 33441 \text{ rupees.}$$

Without giving all these details, if this mean only is given, wouldn't we make the wrong conclusion that all these households have a monthly income around 30000 rupees? This is almost one and a half times the monthly income of ten of these households.

The purpose of calculating the mean is to reduce a whole collection of numbers to a single number, which gives a general understanding of a situation. But numbers in the collection which are very much less or very much more than others (though few) affect the mean a lot.

In our example, it was a single number very much larger than the first ten which changed the mean so much. Can you think of other instances like this where very small or very large numbers influence the mean to give a wrong impression?

Another average

Let's see how we can compute another average which gives a better overall indication of the monthly income of the 11 households. If we write all the incomes in increasing order and take the middle number, 5 of the households would have income less than this and 5 of them would have more.

Let's write the numbers in order:

16500, 17000, 17500, 18000, 18600, 19500,

21000, 21050, 21700, 22000, 175000

The middle number is 19500. It is called the *median* of these numbers. That is, the median monthly income of the 11 households is 19500 rupees. We can put it like this: of all the 11 households, 5 have monthly income less than 19500 rupees and 5 have more than 19500 rupees. That is, the number of households with income less than the median and the number of households with income more than the median are equal.

What if we take only the first 10 households? If we write incomes in the increasing order, there would be two numbers, 18600 and 19500 at the middle, instead of just one number.

Here also, we must choose the median such that the number of items below it and above it are equal. Any number between 18600 and 19500 would do for this. Usually half their sum is taken as the median.

That is, the median monthly income of the first 10 households is

$$\frac{1}{2}(18600 + 19500) = 19050 \text{ rupees}$$

The median income 19050 rupees, like the mean income 19285 rupees gives a reasonable estimate of the economic status of the first ten households (and there is no great difference between the mean and the median either).

What is important here is that the high income of the eleventh household does not change the median much. Also if we say that the median income of some households is 19050 rupees and that the monthly income of one of these is 21000 rupees, we can conclude that this household is better off than more than half the households considered.

(1) The distance covered by Ahirath in long jump practice are

6.10, 6.20, 6.18, 6.20, 6.25, 6.21, 6.15, 6.10

in metres. Find the mean and median. Why is it that there is not much difference between these?

(2) The table below gives the rainfall during the first week of June 2025 in various districts of Kerala.

District	Rainfall (mm)
Kasaragod	108.7
Kannur	89.4
Kozhikode	74.8
Wayanad	72.0
Malappuram	42.6
Palakkad	35.7
Thrissur	66.4
Ernakulam	73.5
Kottayam	69.1
Idukki	50.5
Pathanamthitta	43.6
Alapuzha	93.1
Kollam	39.0
Thiruvananthapuram	37.5

Calculate the mean and median rainfall in Kerala during this week. Why is the mean less than median ?

(3) Prove that for a set of numbers in arithmetic sequence, the mean and median are equal.

Frequency and median

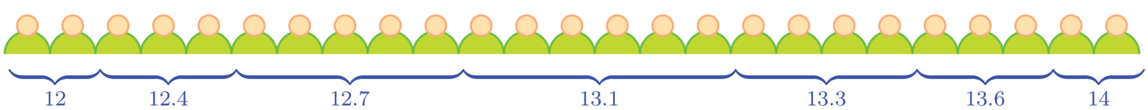
The amount of haemoglobin in blood is usually given as grams per decilitre (that is, 100 millilitres). The table below shows 25 children sorted according to haemoglobin levels, after a blood test.

Haemoglobin (g/dL)	Number of children
12.0	2
12.4	3
12.7	5
13.1	6
13.3	4
13.6	3
14.0	2

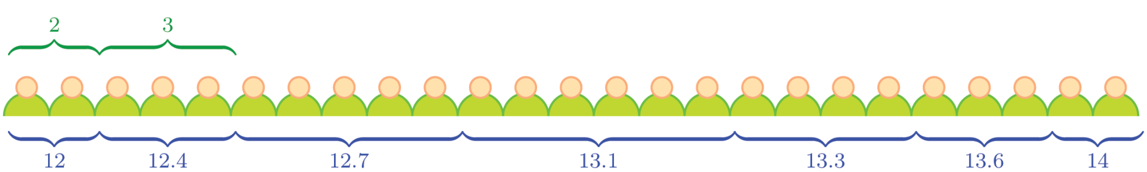
From this, we can compute the mean haemoglobin level. How do we find the median?

The median is that which comes in the middle; that is in this table of 25 children 12 of the children should have haemoglobin level less than the median level and 12 more than the median level.

To find it, we need only make the kids stand in a line, in the order of haemoglobin level and take the level of the thirteenth kid. Imagine the kids standing like this;



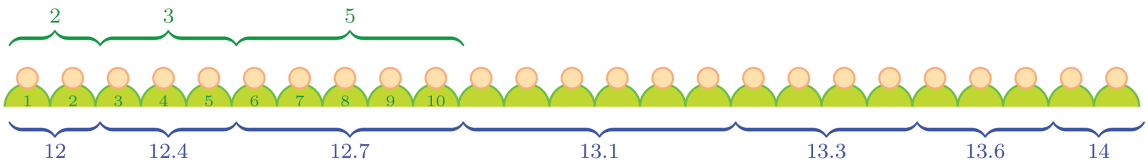
The first two have haemoglobin 12, the next 3 have 12.4 and thus the line grows.



We want the haemoglobin level of the 13th kid. By adding the numbers in the table one by one, we can find his position in the haemoglobin sequence.

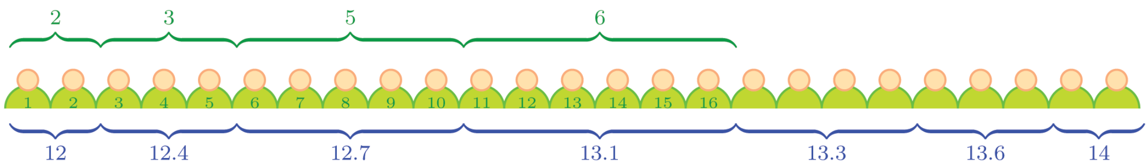
Taking the 2 + 3 = 5 kids of the first two groups, the level rises to 12.4. That is the 5th kid has level 12.4.

Adding the 5 kids in the next group, we have 5 + 5 = 10 kids and the level reaches 12.7.

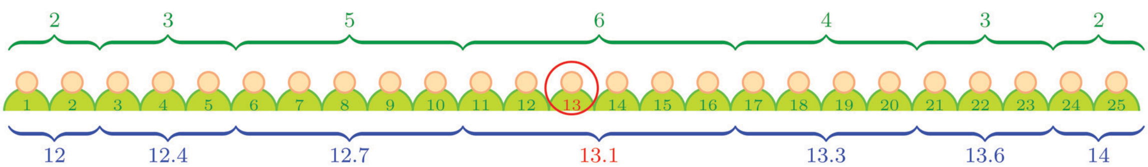


Thus the 10th kid is at level 12.7.

Adding the 6 kids in the next group, we have 10 + 6 = 16 kids.



We need the level of the 13th kid. The level of all the kids from the 11th to the 16th in the line is 13.1. So the 13th kid also has this level, and this is the median level.



We can do this using a table instead of a picture:

Haemoglobin (g/dL)	Number of children
up to 12.0	2
up to 12.4	5
up to 12.7	10
up to 13.1	16
up to 13.3	20
up to 13.6	23
up to 14.0	25

From the table, we can see the haemoglobin level of the kids, from the 11th to the 16th, is 13.1. Since the middle one of 25, that is the 13th, is in this set, the median level can be found as 13.1.

(1) 35 households in a neighbourhood are sorted according to their monthly income in the table below.

Monthly income (Rs)	Number of households
10000	3
11000	7
12000	8
13000	5
14000	5
15000	4
16000	3

Calculate the median income.

(2) The table below shows the workers in a factory sorted according to their daily wages:

Daily wages (Rs)	Number of workers
700	2
800	4
900	5
1000	7
1100	5
1200	4
1300	3

Calculate the median daily wage.

(3) The table below gives the number of babies born in a hospital during a week, sorted according to their birth weight.

Weight (kg)	Number of babies
2.500	4
2.600	6
2.750	8
2.800	10
3.000	12
3.150	10
3.250	8
3.300	7
3.500	5

Calculate the median birth-weight.

Classes and median

The table below shows the workers of a factory sorted according to their wages:

Daily Wages (Rs)	Number of Workers
400 - 500	6
500 - 600	7
600 - 700	10
700 - 800	9
800 - 900	5
900 - 1000	4
Total	41

How do we compute the median daily wage in this factory?

What we want to calculate is the daily wage of the worker in the middle, when the workers are arranged in order from the one earning the least to the one earning the most. Here there are 41 workers in all; so the one in the middle is the 21^{st} in this order.

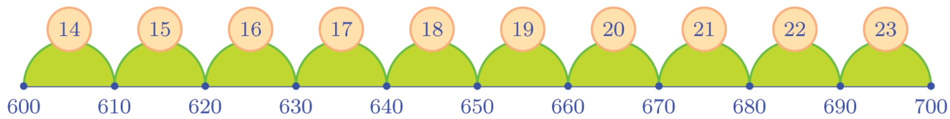
The table splits the wages into different classes. Let's first find the class to which the 21^{st} person belongs. As in our previous problems, let's calculate the total when each class is joined to all classes before it:

Daily Wages (Rs)	Number of Workers
Below 500	6
Below 600	13
Below 700	23
Below 800	32
Below 900	37
Below 1000	41

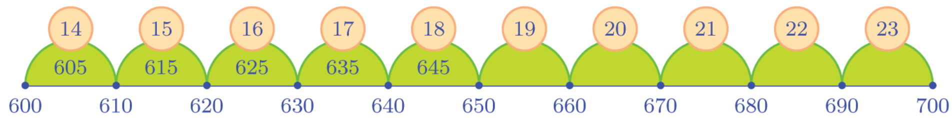
We see from the table that when we take together all those earning less than 600 rupees a day, then we reach up to the 13^{th} person; and when we take together all those earning less than 700 rupees a day, then we reach up to the 23^{rd} person. The 21^{st} person we seek is between them. Thus we find that his daily wage is between 600 and 700 rupees.

What do we do to make this more exact? We only know that the 10 workers from the 14th to the 23rd earn between 600 and 700 rupees; we don't know what the actual earning of each is.

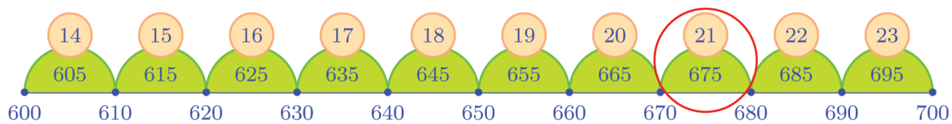
So, we need to make some assumptions. (We made some assumptions when we calculated the mean from a table involving classes and frequencies, remember?). We divide the 100 rupees from 600 to 700 into 10 equal parts and assume that each of these subdivisions contains exactly one person:



We further suppose that the daily wage of each of these workers is the midvalue of the class to which he belongs:



Under these assumptions, we can compute the daily wage of the 21st worker:



We can calculate this without drawing pictures. How did we compute the median wage, on the basis of our assumptions regarding the arrangement of workers?

- The daily wage of the 14th worker is 605 rupees.
- The daily wage of each one thereafter, till the 23rd increases by 10 rupees.
- To reach the 21st worker we want, from the 14th, we must take 7 more workers.

Now it is an arithmetic problem, isn't it?

What number do we get if we start from 605 and add 10 repeatedly 7 times?

We can calculate it as

$$605 + (7 \times 10) = 675$$

Haven't we seen many such problems in the chapter, Arithmetic Sequences?

Let's do one such problem, without drawing pictures. The table below shows the employees in an office, sorted according to their age:

Age	Number of Workers
25 - 30	4
30 - 35	7
35 - 40	8
40 - 45	10
45 - 50	9
50 - 55	8
Total	**46**

We want to compute the median age. Here the total number is 46, which is an even number. So we arrange the employees from the youngest to the oldest and take half the sum of the ages of the 23rd and the 24th persons as the median age. First let's write the cumulated frequencies:

Age	Number of Workers
Below 30	4
Below 35	11
Below 40	19
Below 45	29
Below 50	38
Below 55	46

According to this, the ages of the 10 persons from the 20th to the 29th positions in the order of ages, are between 40 and 45. The persons in the 23rd and 24th position whom we want are in this group.

As in the previous problem, we divide the 5 years between 40 and 45 into 10 equal parts, and assume that each such subdivision contains one person whose age is the midvalue of his class.

Then each subdivision is $\frac{5}{10} = \frac{1}{2}$ year. The age of the 20th person is the midvalue of 40 and $40\frac{1}{2}$, which is $40\frac{1}{4}$. Our assumption means the age of each succeeding person increases by $\frac{1}{2}$ year (till the 29th). So the age of the 23rd person is

$$40\frac{1}{4} + \left(3 \times \frac{1}{2}\right) = 40\frac{1}{4} + 1\frac{1}{2} = 41\frac{3}{4} \text{ years}$$

and the age of the 24th person is

$$41\frac{3}{4}+\frac{1}{2}=42\frac{1}{4} \text{ years}$$

Now to get the median age, we take half the sum of these two:

$$\frac{1}{2}\left(41\frac{3}{4}+42\frac{1}{4}\right)=\frac{1}{2}\times 84=42$$

So the median age is 42. In this example, the persons in the 23rd and 24th positions are in the same class 40 – 45. In contexts where this is not so, we compute the median in a slightly different manner. See this problem.

The table below shows children of a class sorted according to their marks in an exam:

Marks	Number of Children
0 - 10	4
10 - 20	7
20 - 30	9
30 - 40	12
40 - 50	8
Total	**40**

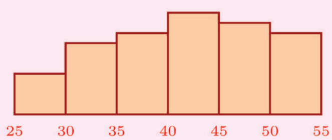
We want to compute the median mark.

First let's write the cumulated frequencies:

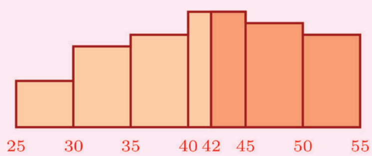
Marks	Number of Children
Below 10	4
Below 20	11
Below 30	20
Below 40	32
Below 50	40

Median area

Do you remember drawing histograms of frequency tables? The histogram of the age problem is like this:



In this, the vertical line through the median splits the picture into two parts:



It is not difficult to see that the areas of the two parts are equal (try it!)

Does the median have this property in all such computations?

Here, if we arrange the children from the one with the least mark to the one with the greatest, then the number of children with marks below 30 is 20, which is exactly half the total number of 40 children.

In this case, we take 30 itself as the median mark. The justification for this is that half the total number of children have marks below 30 and half of them have marks 30 or more.

(1) The table shows some households sorted according to their usage of electricity:

Electricity usage (units)	Number of households
220 - 240	3
240 - 260	6
260 - 280	7
280 - 300	10
300 - 320	9
320 - 340	4

Calculate the median usage of electricity.

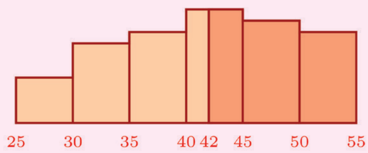
(2) The table below shows the children in a class sorted according to their marks in the math exam:

Marks	No. of Children
0 - 10	4
10 - 20	8
20 - 30	10
30 - 40	9
40 - 50	5

Calculate the median mark of the class.

Median probability

We have noted that the perpendicular through the median divides the histogram into two parts of equal area:



So, if we mark a point on this picture, the probability of it falling on either part is the same (that is the probability is $\frac{1}{2}$).

This means that in the office, given in the problem, if we select someone without any special consideration, the probability of his age being less than 42 or more than 42 is the same.

(3) The table below gives the details of the income tax paid by the employees in an office in a year:

Income tax (Rs)	Number of employees
1000 - 2000	8
2000 - 3000	10
3000 - 4000	15
4000 - 5000	20
5000 - 6000	22
6000 - 7000	8
7000 - 8000	6
8000 - 9000	3

Calculate the median income tax paid.

Notes

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CONSTITUTION OF INDIA

Part IV A

FUNDAMENTAL DUTIES OF CITIZENS

ARTICLE 51 A

Fundamental Duties- It shall be the duty of every citizen of India:

(a) to abide by the Constitution and respect its ideals and institutions, the National Flag and the National Anthem;

(b) to cherish and follow the noble ideals which inspired our national struggle for freedom;

(c) to uphold and protect the sovereignty, unity and integrity of India;

(d) to defend the country and render national service when called upon to do so;

(e) to promote harmony and the spirit of common brotherhood amongst all the people of India transcending religious, linguistic and regional or sectional diversities; to renounce practices derogatory to the dignity of women;

(f) to value and preserve the rich heritage of our composite culture;

(g) to protect and improve the natural environment including forests, lakes, rivers, wild life and to have compassion for living creatures;

(h) to develop the scientific temper, humanism and the spirit of inquiry and reform;

(i) to safeguard public property and to abjure violence;

(j) to strive towards excellence in all spheres of individual and collective activity so that the nation constantly rises to higher levels of endeavour and achievements;

(k) who is a parent or guardian to provide opportunities for education to his child or, as the case may be, ward between age of six and fourteen years.

CHILDREN'S RIGHTS

Dear Children,

Wouldn't you like to know about your rights? Awareness about your rights will inspire and motivate you to ensure your protection and participation, thereby making social justice a reality. You may know that a commission for child rights is functioning in our state called the ***Kerala State Commission for Protection of Child Rights.***

Let's see what your rights are:

- Right to freedom of speech and expression.
- Right to life and liberty.
- Right to maximum survival and development.
- Right to be respected and accepted regardless of caste, creed and colour.
- Right to protection and care against physical, mental and sexual abuse.
- Right to participation.
- Protection from child labour and hazardous work.
- Protection against child marriage.
- Right to know one's culture and live accordingly.
- Protection against neglect.
- Right to free and compulsory education.
- Right to learn, rest and leisure.
- Right to parental and societal care, and protection.

Major Responsibilities

- Protect school and public facilities.
- Observe punctuality in learning and activities of the school.
- Accept and respect school authorities, teachers, parents and fellow students.
- Readiness to accept and respect others regardless of caste, creed or colour.

Contact Address:

Kerala State Commission for Protection of Child Rights

'Sree Ganesh', T. C. 14/2036, Vanross Junction

Kerala University P. O., Thiruvananthapuram - 34, Phone : 0471 - 2326603

Email: childrights.cpcr@kerala.gov.in, rte.cpcr@kerala.gov.in

Website : www.kescpcr.kerala.gov.in

Child Helpline - 1098, Crime Stopper - 1090, Nirbhaya - 1800 425 1400

Kerala Police Helpline - 0471 - 3243000/44000/45000

*Online **R. T. E Monitoring : www.nireekshana.org.in***